PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Memoir on the Theory of the Partition of Numbers. Part I

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XVI. *Memoir on the Theory of the Partition of Numbers.*—Part I.

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§ 1.

Art. 1. I have under consideration multipartite numbers as defined in a former paper.*

I recall that the multipartite number

$$\overline{\alpha\beta\gamma\ldots},$$

may be regarded as specifying $\alpha + \beta + \gamma + \ldots$ things, α of one sort, β of a second, γ of a third, and so forth. If the things be of m different sorts the number is said to be multipartite of order m or briefly an m-partite number. It is convenient to call $\alpha, \beta, \gamma, \ldots$ the first, second, third ... figures of the multipartite number. If such a number be divided into parts each part is regarded as being m-partite; if the order in which the parts are written from left to right is essential we obtain a composition of the multipartite number; whereas if the parts themselves are alone specified, and not the order of arrangement, we have a partition of the multipartite number. This, and much more, is explained in the paper quoted, which is concerned only with the compositions of multipartite numbers.

Art. 2. The far more difficult subject of partitions is taken up in the present paper.

The compositions admitted of easy treatment by a graphical process. An m-partite reticulation or lattice is taken to be the graph of an m-partite number, and on this graph every composition can be satisfactorily depicted.

A suitable graphical representation of the partitions appears to be difficult of attainment. As the Memoir proceeds, the extent to which the difficulties have been overcome will appear. There are several bonds of connection between partitions and compositions; in general, these do not exist between m-partite partitions and m-partite compositions, but arise from a general survey of the partitions and compositions of multipartite numbers of all orders.

These bonds are of considerable service in the gradual evolution of a theory of partitions. Two bonds have already been made known (*loc. cit.*). They both have

* "Memoir on the Theory of the Composition of Numbers," 'Phil. Trans.,' R.S. of London, vol. 184 (1893), A, pp. 835–901.

reference to the perfect partitions of unipartite numbers.* Firstly, there is a one-to-one correspondence between the compositions of the multipartite

$$\overline{\alpha\beta\gamma \ldots},$$

and the perfect partitions of the unipartite

$$a^{\alpha}b^{\beta}c^{\gamma} \ldots - 1;$$

$a, b, c, \ldots$ being any different primes.

Secondly, there is a one-to-one correspondence between the compositions of the unipartite number m and the perfect partitions, comprising m parts, of the whole assemblage of unipartite numbers.

These bonds are interesting but, for present purposes, trivial. We require some correspondence concerning partitions which are not subject to the restriction of being *perfect*.

Art. 3. I first proceed to explain an important link connecting unipartite with multipartite partitions arising from the notion of the separation of the partition of a number (whether unipartite or multipartite) into separates; a notion which leads to a theory of the separations of a partition of a number* which was partially set forth in the series of Memoirs referred to in the foot-note. The theory of separations arises in a perfectly natural manner in the evolution of the theory of symmetrical algebra, and is, I venture to think, of considerable algebraical importance. Up to the present time the theory has been worked out as it was required for algebraical purposes; the various definitions and theorems are scattered about several Memoirs in a manner which is inconvenient for reference, and it therefore will be proper, while explaining the connection with compositions, to bring the salient features of the theory together under the eye of the reader. It suffices for the most part to deal with unipartite numbers.

The Theory of Separations.

Definitions.

Art. 4. A number is partitioned into parts by writing down a set of positive numbers (it is convenient, but not necessary to assume positive parts, and occasionally to regard zero as a possible part) which, when added together, reproduce the original number.

The constituent numbers, termed parts, are written in descending numerical order from left to right and are usually enclosed in a bracket ( ). This succession of

* See also "The Perfect Partitions of Numbers and the Compositions of Multipartite Numbers." The Author, 'Messenger of Mathematics,' New Series, No. 235, November, 1890.

numbers is termed a partition of the original number; and this number, *quâ* partitions, is termed by SYLVESTER the partible number.

A partition of a number is separated into separates by writing down a set of partitions, each in its own brackets, such that when all the parts of the partitions are assembled in a single bracket and arranged in order, the partition which is separated is reproduced. The constituent partitions, which are the separates, are written down from left to right in descending numerical order as regards the weights of the partitions.

N.B. The partition (pqr) is said to have a weight $p+q+r$.

The partition separated may be termed the separable partition.

Taking as separable partition

$$(p_1p_2p_3p_4p_5),$$

two separations are

$$(p_1p_2)\,(p_3p_4)\,(p_5),$$
$$(p_1p_2p_3)\,(p_4p_5),$$

and there are many others.

If the successive weights of the separates be

$$w_1,\ w_2,\ w_3,\ \ldots,$$

the separation is said to have a specification

$$(w_1,\ w_2,\ w_3,\ \ldots);$$

the specification being denoted by a partition of the weight w of the separable partition.

The *degree* of a separation is the sum of the highest parts of the several separates.

If the separation be

$$(p_1\ldots)^{j_1}\,(p_s\ldots)^{j_2}\,(p_t\ldots)^{j_3}\ldots$$

the *multiplicity* of the separation is defined by the succession of indices

$$j_1,\ j_2,\ j_3,\ \ldots$$

The characteristics of a separation are

(i.) The weight.
(ii.) The separable partition.
(iii.) The specification.
(iv.) The degree.
(v.) The number of separates.
(vi.) The multiplicity.

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Art. 5. The separations of a given partition may be grouped in a manner which is independent of their specifications.

Consider the separable partition

$$(p_1^3\, p_2^2),$$

which is itself to be regarded as one amongst its own separations.

Viewed thus, it has *quâ* partition a multiplicity (32).

Write down any one of its separations, say

$$(p_1^2)\ (p_1 p_2)\ (p_2).$$

This separation may be regarded as being compounded of the two separations

$$(p_1^2)\ (p_1) \text{ of } (p_1^3)$$

and

$$(p_2)^2 \text{ of } (p_2^2).$$

Three other separations enjoy the same property, viz.,

$$(p_1^2)\ (p_1)\ (p_2)^2,$$
$$(p_1^2\, p_2)\ (p_1)\ (p_2),$$
$$(p_1^2\, p_2)\ (p_1\, p_2)\,;$$

for, on suppressing p_2 in each, we are left with

$$(p_1^2)\ (p_1)\,;$$

and on suppressing p_1 there remains

$$(p_2)^2.$$

These four separations

$$\left.\begin{array}{l}(p_1^2)\ (p_1\, p_2)\ (p_2)\\ (p_1^2)\ (p_1)\ (p_2)^2\\ (p_1^2\, p_2)\ (p_1)\ (p_2)\\ (p_1^2\, p_2)\ (p_1\, p_2)\end{array}\right\}\ \text{Set } \{(21), (1^2)\},$$

form a set which is defined by two partitions, one appertaining to each of the two numbers which define the multiplicity of the separable partition.

Thus the first number 3 of the multiplicity occurs in each separation of the group in the partition (21), and the second number 2 in the partition (1^2).

There are as many sets of separations as there are combinations of a partition of 3 with a partition of 2.

In the present instance there are six sets, viz.,

$$S\{(3),\ (2)\},$$
$$S\{(3),\ (1^2)\},$$
$$S\{(21),\ (2)\},$$
$$S\{(21),\ (1^2)\},$$
$$S\{(1^3),\ (2)\},$$
$$S\{(1^3),\ (1^2)\}.$$

In general, if

$$(p_1^{\pi_1}\, p_2^{\pi_2}\, p_3^{\pi_3} \ldots)$$

be the separable partition, the multiplicity is

$$(\pi_1\ \pi_2\ \pi_3 \ldots);$$

and if the unipartite π_s possess ρ_s partitions, the separations can be arranged in

$$\rho_1\ \rho_2\ \rho_3 \ldots$$

sets.

I observe that the notion of sets of separations enters in a fundamental manner into the theory of symmetric functions.

Art. 6. One of the first problems encountered in the arithmetical theory is the enumeration of the separations of a given partition. I shall prove that the number of separations of the partition

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots)$$

is identical with the number of partitions of the multipartite number

$$(\overline{\pi_1\pi_2\pi_3 \ldots}),$$

formed from the multiplicity of the separable partition.*

If we separate $(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3} \ldots)$ so that one separate of the separation is

$$(q_1^{\chi_1}q_2^{\chi_2}q_3^{\chi_3} \ldots),$$

it is clear that we can partition the multipartite $(\overline{\pi_1\pi_2\pi_3 \ldots})$ in such wise that one part of the partition is

$$(\overline{\chi_1\chi_2\chi_3 \ldots}).$$

* Note that the rule ———— distinguishes a multipartite number from a partition of a unipartite number.

It is also manifest that to each separate of the separation corresponds a part of the partition, and that we obtain a partition

$$(\overline{\chi_1\chi_2\chi_3\ldots} \quad \ldots \quad \ldots)$$

of the multipartite

$$(\overline{\pi_1\pi_2\pi_3\ldots}),$$

in correspondence with each separation

$$(q_1^{\chi_1}q_2^{\chi_2}q_3^{\chi_3}\ldots)\ (\ldots)\ (\ldots)\ldots$$

of the partition

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots).$$

There is, therefore, identity of enumeration. Also, the enumeration of the separations into k separates is identical with that of the partitions into k parts.

Ex. gr., the subjoined correspondence :—

Separations.	Partitions.
(p^2q^2)	$(\overline{22})$,
$(p^2)\ (q^2)$,	$(\overline{20}\ \overline{02})$,
$(p^2q)\ (q)$,	$(\overline{21}\ \overline{01})$,
$(p)\ (pq^2)$,	$(\overline{10}\ \overline{12})$,
$(pq)^2$,	$(\overline{11}^2)$,
$(p^2)\ (q)^2$,	$(\overline{20}\ \overline{01}^2)$,
$(p)^2\ (q^2)$,	$(\overline{10}^2\ \overline{02})$,
$(pq)\ (p)\ (q)$,	$(\overline{11}\ \overline{10}\ \overline{01})$,
$(p)^2\ (q)^2$,	$(\overline{10}^2\ \overline{01}^2)$.

Art. 7. It is important to take note of the fact that the subject of the separations of partitions of unipartite numbers necessitates the consideration of the partitions of multipartite numbers.

The partitions of a multipartite number are divisible into sets in the same manner as the separations of a unipartite number. In the case of the number

$$(\pi_1\pi_2\pi_3\ldots)$$

there are $\rho_1\rho_2\rho_3\ldots$ sets, where ρ_s is the number of partitions of the unipartite π_s.

The same succession of numbers may be employed to denote either a multipartite number or the partition of a unipartite number, so that we naturally find great similarity between the theories of unipartite partitions and of multipartite numbers. We see above that the separations of partitions of unipartites are in co-relation with the partitions of multipartites.

Art. 8. In a natural manner the separations of a partition of a multipartite present themselves for consideration. In correspondence we find what may be termed a double separation of a partition of a unipartite.

Ex. gr. Consider the partition $(\overline{20}\ \overline{01}^2)$ of the multipartite $(\overline{22})$, which is co-related to the separation $(p^2)(q)^2$ of the partition (p^2q^2) of the unipartite $2p + 2q$.

Of this partition we find a separation

$$(\overline{20}\ \overline{01})\ (\overline{01})$$

in correspondence with a *double* separation

$$\{(p^2)\ (q),\ (q)\}$$

of the partition (p^2q^2).

Hence the enumeration of the separations of a multipartite partition is identical with that of the double separations of a unipartite partition.

In general n-tuple separations of unipartite numbers correspond one-to-one with $n - 1$-tuple separations of multipartite numbers.

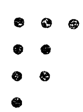
For the present I leave the subject of separations, merely remarking that the theory was made the basis of all the memoirs on symmetric functions to which reference has been given, and that algebraically considered they are of extraordinary interest.

§ 2. The Graphical Representation of Partitions.

Art. 9. In an important contribution to the theory of unipartite partitions Sylvester* adopted a graphical method which threw great light on the subject, and was fruitful in algebraical results.

The method consisted in arranging rows of nodes, each row corresponding to a part of the partition and containing as many nodes as the number expressing the magnitude of the part.

Ex. gr., the partition (32^21) has the graph



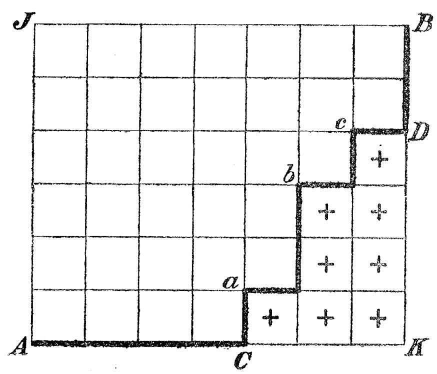
* "A Constructive Theory of Partitions," by J. J. Sylvester, with insertions by Dr. F. Franklin, 'American Journal of Mathematics,' vol. 5.

This method cannot be simply extended to the case of multipartite partitions, though some progress can be made in this direction as will be shown.

Art. 10. SYLVESTER's graphs naturally present themselves in the graphical representation of the compositions of bipartite numbers.

As the graph of a bipartite $(\overline{pq})$ we take $p + 1$ lines parallel and at equal distances apart and cut them by $q + 1$ other lines at equal distances apart and at right angles to the former; (N.B. The right angle is not essential,) thus forming a reticulation or lattice.

The figure represents the reticulation of the bipartite $(\overline{76})$.



I recall that A, B are the initial and final points of the graph, and that the remaining intersections are termed the "points" of the graph.

The lines of the graph have either the direction AK (called the α direction) or the direction AJ (called the β direction).

Each line is made up of segments, and we speak of α-segments and of β-segments, indicating that the lines, on which lie the segments, are in the α and β directions.

If the bipartite $(\overline{pq})$ have a composition

$$(\overline{p_1q_1}\ \overline{p_2q_2}\ \overline{p_3q_3}\ \ldots),$$

the composition is delineated upon the graph, as follows :—

Starting from the point A we pass over p_1 α-segments and then over q_1 β-segments, and place a node at the point arrived at. Starting again from this node, we pass over p_2 α-segments and q_2 β-segments, and place a second node at the point then reached; we proceed similarly with the other parts of the composition until finally the point B is reached. At this point it is not necessary to place a node. In this manner the composition containing θ parts is represented by $\theta - 1$ nodes placed at $\theta - 1$ different "points" of the graph.

The segments, passed over in tracing the composition, form a line of route through the reticulation. In general many compositions have the same line of route. Along every line of route there are $p + q - 1$ "points," which *may* be nodes. A certain number of these points *must* be nodes. These occur at all points where there is a change from the β to the α direction. They are termed "essential nodes."

In the line of route traced in the figure the points a, b, c are essential nodes.

Along a line of route there is a composition which is depicted by the essential nodes alone. This is termed the "principal composition along the line of route."

In the figure the principal composition is

$$(\overline{41}\ \overline{12}\ \overline{11}\ \overline{12}).$$

I have shown (*loc. cit.*) that the number of lines of route which possess s essential nodes is

$$\binom{p}{s}\binom{q}{s};$$

and that the total number of lines of route is

$$\sum_{s=0}^{s=q}\binom{p}{s}\binom{q}{s} = \binom{p+q}{p} \qquad (p \geq q).$$

We remark that the line of route divides the reticulation into two portions, an upper portion AJBDC, and a lower portion CKD.

Placing a Sylvester-node in each square of the lower portion, we recognise at once SYLVESTER's regularised graph of the partition

$$(32^21)$$

of the unipartite number 8.

Similarly, from the upper portion, we obtain SYLVESTER's regularised graph of the partition

$$(7^265^24),$$

of the unipartite number 34 (thirty-four).

Whatever be the line of route, we simultaneously exhibit two of SYLVESTER's regularised graphs, one of a partition of the unipartite N, and one of a partition of the unipartite $pq - \mathrm{N}$.

The two partitions may be termed complementary in respect of the unipartite number pq.

Art. 11. This interesting bond between the partitions of unipartites and the compositions of bipartites, I propose to submit to a detailed examination. The partitions, with which we are concerned, are limited in magnitude of part to p, and in number of parts to q.

Moreover (as in the bipartite), we may suppose the numbers p, q, to be interchanged. This would simply amount to rotating the reticulation through a right angle.

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The line of route partitions the reticulation into two parts, each of which may be regarded as a partition of a unipartite. In fact, a line of route graphically represents a pair of partitions.

These partitions can be equally depicted by the essential nodes only that occur along the line of route. It will be more convenient, for some purposes, to take the line of route, and not merely the essential nodes, to be the graphical representation. Attention for the present will be limited to the partition lying to the North-West of the line of route (*i.e.*, towards the point J). A line of route involves bends ⌋, termed "left-bends," and bends ⌈, termed "right-bends." Essential nodes occur at the angular points of the latter. The North-West partition has as many *different* parts as there are left-bends on the line of route. The number of lines of route which have s left-bends is equal to the number which have s right-bends. This may be seen by rotating the reticulation through two right angles.

This number is $\binom{p}{s}\binom{q}{s}$ (*loc. cit.*, Art. 22).

Hence :—

"The number of partitions of all numbers into s different parts limited in magnitude to p and in number to q is

$$\binom{p}{s}\binom{q}{s}."$$

Art. 12. The number of different partitions is equal to the number of lines of route, and this is

$$\sum_s \binom{p}{s}\binom{q}{s} = \binom{p+q}{p} \qquad (p \not< q).$$

Hence :—

"The number of partitions, of all numbers, into parts limited in magnitude to p, and in number to q is

$$\binom{p+q}{p}."$$

Art. 13. A line of route, with s left-bends, has either $s-1$, s or $s+1$ right-bends. If it commences by tracing an α-segment and ends by tracing a β-segment, the number is $s-1$. If it commences by tracing an α-segment and ends by tracing an α-segment, the number is s. If a β-segment begins and a β-segment ends the line of route, the number is s, and if a β-segment begins and an α-segment ends, the number is $s+1$.

If an α-segment begins the line, the North-West partition has exactly q parts. If an α-segment ends, the highest part is less than p. If a β-segment begins, the number of parts is less than q, and if a β-segment ends, the highest part is equal to p.

Inspection of the Memoir cited shows that the enumeration of the lines of route possessing s left-bends and $s-1$, s, $s+1$ right-bends respectively are given by

$$\binom{p-1}{s-1} \quad \binom{q-1}{s-1},$$

$$\binom{p-1}{s}\binom{q-1}{s-1}+\binom{p-1}{s-1}\binom{q-1}{s},$$

$$\binom{p-1}{s}\binom{q-1}{s};$$

hence :—

"The number of partitions of all numbers which have exactly q parts, a highest part equal to p, and s different parts is

$$\binom{p-1}{s-1}\binom{q-1}{s-1}."$$

"The number of partitions of all numbers which have exactly q parts, a highest part less than p, and s different parts, *or* which have less than q parts, a highest part equal to p, and s different parts is

$$\binom{p-1}{s}\binom{q-1}{s-1}+\binom{p-1}{s-1}\binom{q-1}{s}."$$

"The number of partitions of all numbers which have less than q parts, a highest part less than p, and s different parts is

$$\binom{p-1}{s}\binom{q-1}{s}."$$

Ex. gr. If $p = q = 3$, $s = 2$ the partitions enumerated by these three theorems are

$$(31^2),\quad (3^21),\quad (32^2),\quad (3^22)$$
$$(31),\quad (21^2),\quad (32),\quad (2^21)$$
$$(21)$$

respectively.

Art. 14. It is clear that all identical relations between binomial coefficients yield results in this theory of partitions. *Ex. gr.*, such relations as

$$\binom{p+q}{p}-\binom{p+q-1}{p-1}=\binom{p+q-1}{p}$$

$$\binom{p-1}{s}\binom{q-1}{s}-\binom{p-2}{s}\binom{q-1}{s}=\binom{p-2}{s-1}\binom{q-1}{s}$$

admit of immediate interpretation.

Art. 15. A line of route has in general right and left bends. The whole number of bends may be even or uneven. To determine the number of lines, with a given number of bends, we must separate the two cases. If the bends be $2k$ in number, k must be right and k left and the enumeration gives

$$\binom{p-1}{k}\binom{q-1}{k-1}+\binom{p-1}{k-1}\binom{q-1}{k}.$$

If the bends be $2k+1$ in number we may have k right-bends and $k+1$ left-bends or $k+1$ right and k left. The enumeration gives

$$2\binom{p-1}{k}\binom{q-1}{k}.$$

Hence the number of pairs of complementary partitions which have each k different parts is

$$\binom{p-1}{k}\binom{q-1}{k-1}+\binom{p-1}{k-1}\binom{q-1}{k};$$

and the number of pairs in which the partitions have k and $k+1$ different parts respectively, is

$$2\binom{p-1}{k}\binom{q-1}{k}.$$

Adding the number of lines of route with $2k-1$ bends to the number with $2k$ bends we obtain the number

$$\frac{p+q}{k}\binom{p-1}{k-1}\binom{q-1}{k-1}.$$

Hence the identity,

$$(p+q)\sum_k \frac{1}{k}\binom{p-1}{k-1}\binom{q-1}{k-1}=\binom{p+q}{p};$$

or

$$\binom{p-1}{0}\binom{q-1}{0}+\tfrac{1}{2}\binom{p-1}{1}\binom{q-1}{1}+\tfrac{1}{3}\binom{p-1}{2}\binom{q-1}{2}+\ldots=\frac{(p+q-1)!}{p!\,q!};$$

which may be established independently.

Art. 16. Consider the lines of route which pass through a particular point P of the graph, say the point distant from A, a α-segments and b β-segments. In the reticulation AP we may draw any line of route, and any line also in the reticulation PB. In AB we can thus obtain

$$\binom{a+b}{a}\binom{p+q-a-b}{p-a}$$

lines of route passing through P.

In the associated North-West partitions, the highest part $\not< a$ and the number of parts $\not< q-b$.

Put $q-b=c$, and we obtain the theorem :—

"Of the partitions whose parts are limited in magnitude to p and in number to q, there are,

$$\binom{q+a-c}{a}\binom{p-a+c}{c},$$

such that the highest part $\not< a$ and number of parts $\not< c$."

In particular, if

$$a+b=p,$$

or

$$a-c=p-q,$$

the number is

$$\binom{p}{p-a}\binom{q}{p-a};$$

which enumerates the lines of route with $p-a$ left-bends. There is thus a one-to-one correspondence between the lines of route passing through the point for which $(a, b) = (a, p-a)$, and the lines of route with $p-a$ left-bends, and hence also between the partitions under consideration and those which involve parts of $p-a$ different kinds.

Art. 17. A line of route is a graphical representation of a principal composition of the bipartite. Such a composition being

$$(\overline{p_1q_1}\ \overline{p_2q_2}\ \overline{p_3q_3} \ldots \overline{p_sq_s}),$$

the numbers

$$q_1,\ p_2,\ q_2,\ p_3,\ q_3 \ldots p_s$$

are all superior to zero. A positive number q_1 of the bipart $\overline{p_1q_1}$ is adjacent to a positive number p_2 of the bipart $\overline{p_2q_2}$, and we may assert of the composition that all its contacts are positive-positive (see Art. *loc. cit.*).

Each line of route represents a composition with positive-positive contacts, and there is a one-to-one correspondence. Hence:

"There is a one-to-one correspondence between the compositions, with positive-positive contacts, of the bipartite $\overline{pq}$ and the partitions of all unipartite numbers into parts limited in magnitude by p and in number by q."

Further, there are as many such compositions with $s+1$ parts as there are such partitions which involve s *different* parts.

Art. 18. Every line of route through the reticulation of $\overline{pq}$ may be represented by a permutation of the letters in $\alpha^p\beta^q$. We have merely to write down the α and β segments as they occur along the line of route to obtain such a permutation.

The composition

$$(\overline{p_1q_1}\ \overline{p_2q_2}\ \overline{p_3q_3} \ldots \overline{p_{s+1}\ q_{s+1}}),$$

gives the permutation

$$\alpha^{p_1}\beta^{q_1}\alpha^{p_2}\beta^{q_2}\alpha^{p_3}\beta^{q_3}\ldots\alpha^{p_{s+1}}\beta^{q_{s+1}}.$$

From what has gone before it will be seen that every permutation of $\alpha^p\beta^q$ corresponds to a partition of a unipartite number into parts limited in magnitude to p and in number to q. Every theorem in permutations of two *different* letters will thus yield a theorem in partitions of unipartite numbers.

The North-West partition associated with the above-written permutation is easily seen to be (writing the parts in ascending order as regards magnitude, viz.: in SYLVESTER's regularised orders reversed)

$$\left(p_1^{q_1}\,\overline{p_1+p_2}^{q_2}\,\overline{p_1+p_2+p_3}^{q_3}\ldots\overline{p_1+p_2+\ldots+p_{s+1}}^{q_{s+1}}\right).$$

This is a partition of the unipartite

$$q_1p_1+q_2(p_1+p_2)+q_3(p_1+p_2+p_3)+\ldots+q_{s+1}(p_1+p_2+\ldots+p_{s+1})$$

into $q_1+q_2+q_3+\ldots+q_{s+1}$ parts, the highest part being $p_1+p_2+p_3+\ldots+p_{s+1}$.

If p_1 be zero there will be less than q parts. If q_{s+1} be zero the highest part will be less than p. On the other hand if p_1 be not zero there are exactly q parts, and if q_{s+1} be not zero the highest part is p.

Art. 19. Observe that we have a fourfold correspondence, viz., between

(1.) The lines of route in the reticulation of the bipartite $\overline{pq}$.
(2.) The compositions, with positive-positive contacts, of the bipartite number $\overline{pq}$.
(3.) The permutations of the letters forming the product $\alpha^p\beta^q$.
(4.) The partitions of all unipartite numbers into parts limited in magnitude to p and in number to q.

And also, in particular, between—

(1.) The lines of route with s right-bends or with s left-bends.
(2.) The compositions into $s+1$ parts.
(3.) The permutations with s, $\alpha\beta$ or with s, $\beta\alpha$ contacts.
(4.) The partitions involving s *different* parts.

Art. 20. The generating function for the number of lines of route through the reticulation which possess s left-bends is

$$1+\binom{p}{1}\binom{q}{1}\mu+\binom{p}{2}\binom{q}{2}\mu^2+\ldots+\binom{p}{s}\binom{q}{s}\mu^s+\ldots+\binom{p}{q}\binom{q}{p}\mu^q$$

which is the coefficient of $\alpha^p\beta^q$ in the product

$$(\alpha+\mu\beta)^p\;(\alpha+\beta)^q;$$

(see *loc. cit.*, Art. 24),

and this is the coefficient of $\alpha^p\beta^q$ in the development of the fraction

$$\frac{1}{1-\alpha-\beta+(1-\mu)\,\alpha\beta};$$

(see "Memoir on a Certain Class of Generating Functions in the Theory of Numbers," 'Phil. Trans.,' Roy. Soc. of London, vol. 185 (1894), A, pp. 111–160), and this fraction may be written

$$\sum_0 \frac{\alpha^s\beta^s}{(1-\alpha)^{s+1}(1-\beta)^{s+1}}\mu^s.$$

Hence

$$\frac{\alpha^s\beta^s}{(1-\alpha)^{s+1}(1-\beta)^{s+1}}$$

is the generating function for the lines of route in all bipartite reticulations which possess s left-bends *or* s right-bends, and also for the other entities in correspondence therewith. In particular it enumerates all unipartite partitions into s *different* parts limited, in any desired manner, in regard to number and magnitude.

Art. 21. Various theorems in algebra are derivable from the foregoing theorems.

The generating function for the partitions of all unipartite numbers into parts limited in magnitude to p and in number to q is

$$\frac{1}{1-a\,.\,1-x\,.\,1-ax\,.\,1-ax^2\,.\,1-ax^3\ldots\ldots1-ax^p};$$

the enumeration being given by the coefficients of a^qx^{pq} in the ascending expansion.

The G.F. is redundant as we are only concerned with that portion, of the expanded form, which proceeds by powers of ax^p.

The foregoing theory enables us to isolate this portion, inasmuch as we know it to have the expression

$$1+\binom{p+1}{1}ax^p+\binom{p+2}{2}a^2x^{2p}+\binom{p+3}{3}a^3x^{3p}+\ldots\ldots+\binom{p+q}{p}a^qx^{qp}+\ldots\ldots$$

which may be written

$$(1-ax^p)^{-p-1}.$$

As a verification of the simplest cases we find the identities

$$\frac{1}{1-a\,.\,1-x\,.\,1-ax}=\frac{1}{(1-ax)^2}\left(1+\frac{a}{1-a}+\frac{x}{1-x}\right)$$

$$\frac{1}{1-a\,.\,1-x\,.\,1-ax\,.\,1-ax^2}=\frac{1}{(1-ax^2)^3}\left\{1+\frac{a}{1-a}+\frac{x}{1-x}+\frac{ax}{1-a}+\frac{ax\,(1-ax^2)}{1-a\,.\,1-ax}\right\},$$

a simple inspection of which demonstrates the validity of the theorem in these cases

To obtain a general formula write

$$\frac{(1-ax^p)^{p+1}}{1-a\,.\,1-x\,.\,1-ax\,.\,1-ax^2\ldots 1-ax^p\,.}$$

$$=\frac{(1-ax^{p-1})^p}{1-a\,.\,1-x\,.\,1-ax\,.\,1-ax^2\ldots 1-ax^{p-1}}+\mathrm{U}_p(x)\,;$$

then

$$\mathrm{U}_p(x)=\frac{(1-ax^p)^{p+1}-(1-ax^p)\,(1-ax^{p-1})^p}{1-a\,.\,1-x\,.\,1-ax\,.\,.\,1-ax^2\ldots 1-ax^p}$$

$$=\frac{ax^{p-1}\{(1-ax^p)^{p-1}+(1-ax^p)^{p-2}\,(1-ax^{p-1})+\ldots+(1-ax^{p-1})^{p-1}\}}{1-a\,.\,1-ax\,.\,1-ax^2\ldots 1-ax^{p-1}}$$

and we have in succession

$$\mathrm{U}_0(x)=\frac{1}{1-x}\,,$$

$$\mathrm{U}_1(x)=\frac{a}{1-a}\,,$$

$$\mathrm{U}_2(x)=\frac{ax}{1-a}+\frac{ax\,(1-ax^2)}{1-a\,.1-ax}\,,$$

$$\mathrm{U}_3(x)=\frac{ax^2\,(1-ax^2)}{1-a\,.1-ax}+\frac{ax^2\,(1-ax^3)}{1-a\,.1-ax}+\frac{ax^2\,(1-ax^3)^2}{1-a\,.1-ax\,.1-ax^2}\,,$$

$$\mathrm{U}_4(x)=\frac{ax^3\,(1-ax^3)^2}{1-a\,.1-ax\,.1-ax^2}+\frac{ax^3\,(1-ax^3)\,(1-ax^4)}{1-a\,.1-ax\,.1-ax^2}$$

$$+\frac{ax^3\,(1-ax^4)^2}{1-a\,.1-ax\,.1-ax^2}+\frac{ax^3\,(1-ax^4)^3}{1-a\,.1-ax\,.1-ax^2.1-ax^3}\,;$$

and, in general,

$$\mathrm{U}_p(x)=\frac{ax^{p-1}\,(1-ax^{p-1})^{p-2}}{1-a\,.1-ax\ldots 1-ax^{p-2}}+\frac{ax^{p-1}\,(1-ax^{p-1})^{p-3}\,(1-ax^p)}{1-a\,.1-ax\ldots 1-ax^{p-2}}$$

$$+\frac{ax^{p-1}\,(1-ax^{p-1})^{p-4}\,(1-ax^p)^2}{1-a\,.1-ax\,.\,,.1-ax^{p-2}}+\ldots$$

$$+\frac{ax^{p-1}\,(1-ax^p)^{p-2}}{1-a\,.1-ax\ldots 1-ax^{p-2}}+\frac{ax^{p-1}\,(1-ax^p)^{p-1}}{1-a\,.1-ax\ldots 1-ax^{p-2}.1-ax^{p-1}}\,.$$

Hence,

$$\frac{1}{1-x\,.1-a\,.1-ax\,.1-ax^2\ldots 1-ax^p}=\frac{1}{(1-ax^p)^{p+1}}\sum_0^p\mathrm{U}_p(x)\,;$$

a valuable expansion.

Simple inspection of this formula shows that $(1-ax^p)^{-p-1}$ represents that portion of the G.F. which is a function of ax^p only.

Art. 22. Again, the partitions of all unipartite numbers into s *different* parts, limited in magnitude to p and in number to q, are enumerated by the coefficient of $a^q b^s x^{pq}$ in the development of the product

$$\frac{1}{1-x}\cdot\frac{1}{1-a}\left(1+\frac{abx}{1-ax}\right)\left(1+\frac{abx^2}{1-ax^2}\right)\ldots\left(1+\frac{abx^p}{1-ax^p}\right);$$

or by the coefficient of $a^q x^{pq}$ in

$$\frac{a^s}{1-x\,.\,1-a}\Sigma\frac{x^{k_1+k_2+k_3+\ldots+k_s}}{(1-ax^{k_1})(1-ax^{k_2})(1-ax^{k_3})\ldots(1-ax^{k_s})},$$

where $k_1, k_2, k_3, \ldots k_s$ are any s different numbers drawn from the natural series

$$1, 2, 3, \ldots p;$$

and the summation is in respect of all such selections. This is the coefficient of

$$a^{q-s}x^{pq-\binom{s+1}{2}}$$

in

$$\frac{1}{1-x\,.\,1-a}\Sigma\frac{x^{k_1+k_2+k_3+\ldots+k_s-\binom{s+1}{2}}}{(1-ax^{k_1})(1-ax^{k_2})(1-ax^{k_3})\ldots(1-ax^{k_s})}.$$

Taking the former of these two expressions; inasmuch as we know from the reticulation theory that the coefficient of $a^q x^{pq}$ is $\binom{p}{s}\binom{q}{s}$, we find that the effective portion of the generating function is

$$\binom{p}{s}\binom{s}{s}(ax^p)^s+\binom{p}{s}\binom{s+1}{s}(ax^p)^{s+1}+\ldots \textit{ad inf.},$$

which is

$$\binom{p}{s}\frac{(ax^p)^s}{(1-ax^p)^{s+1}};$$

viz., we have succeeded in isolating that portion of the generating function which proceeds by powers of ax^p only.

The latter expression of the generating function also, is seen to have an effective portion

$$\binom{p}{s}\frac{x^{ps-\binom{s+1}{2}}}{(1-ax^p)^{s+1}}.$$

The isolations thus effected would, I believe, be difficult to accomplish algebraically.

Art. 23. Again, regarding p and q as constant and s as variable, we know that the coefficient of $(ax^p)^q$ in the product

$$\frac{1}{1-x\,.\,1-a}\left(1+\frac{abx}{1-ax}\right)\left(1+\frac{abx^2}{1-ax^2}\right)\cdot\;\cdot\left(1+\frac{abx^p}{1-ax^p}\right),$$

is

$$\binom{p}{0}\binom{q}{0}+\binom{p}{1}\binom{q}{1}b+\binom{p}{2}\binom{q}{2}b^2+\ldots+\binom{p}{q}\binom{q}{q}b^q;$$

calling this expression B_q, the effective portion of the generating function is written

$$1+B_1ax^p+B_2(ax^p)^2+B_3(ax^p)^3+\ldots \text{ ad inf.}$$

Art. 24. I recall now the generating function which enumerates the partitions of all unipartite numbers into parts limited in magnitude by p and in number by q, viz. :—

$$\frac{1}{(1-x)(1-a)(1-ax)(1-ax^2)\ldots(1-ax^p)}$$

The coefficient of a^q in this development is well known to be

$$\frac{(1-x^{q+1})(1-x^{q+2})\ldots(1-x^{q+p})}{(1-x)^2(1-x^2)(1-x^3)\ldots(1-x^p)}.$$

Hence the coefficient of a^qx^{pq} in the former is the same as the coefficient of x^{pq} in the latter. This we know to have the value $\binom{p+q}{q}$.

Art. 25. Numerous theorems of *isolation*, in the senses in which the word is employed in this paper, may be obtained from the reticulation theorems. I proceed to give some of those which present features of interest.

A previous result was to the effect that the number of partitions of all numbers which have exactly q parts, a highest part equal to p and s *different* parts is enumerated by

$$\binom{p-1}{s-1}\binom{q-1}{s-1}.$$

Hence, without specification of s, the number is

$$\sum_s\binom{p-1}{s}\binom{q-1}{s}=\binom{p+q-2}{p-1}.$$

This enumeration is also given by the coefficient of $a^{q-1}x^{p(q-1)}$ in the function

$$\frac{1}{(1-x)(1-ax)\ldots(1-ax^p)}$$

Hence we can isolate that portion of the generating function which contains only powers of ax^p. It is

$$1 + \binom{p}{1} ax^p + \binom{p+1}{2} (ax^p)^2 + \binom{p+2}{3} (ax^p)^3 + \ldots\ldots ;$$

or

$$\frac{1}{(1 - ax^p)^p}.$$

This fact leads as before to an expansion theorem.

Putting

$$\frac{(1-ax^p)^p}{1-x \,.\, 1-ax \ldots . 1-ax^p} = \frac{(1-ax^{p-1})^{p-1}}{1-x \,.\, 1-ax \ldots . 1-ax^{p-1}} + \mathrm{V}_p(x),$$

then of course

$$\frac{1}{1-x \,.\, 1-ax \ldots . 1-ax^p} = \frac{1}{(1-ax^p)^p} \sum_{p=1}^{p=p} \mathrm{V}_p(x),$$

and we find in succession

$$\mathrm{V}_1(x) = \frac{1}{1-x},$$

$$\mathrm{V}_2(x) = \frac{ax}{1-ax},$$

$$\mathrm{V}_3(x) = \frac{ax^2}{1-ax} + \frac{ax^2(1-ax^3)}{1-ax \,.\, 1-ax^2}$$

$$\mathrm{V}_4(x) = \frac{ax^3(1-ax^3)}{1-ax \,.\, 1-ax^2} + \frac{ax^3(1-ax^4)}{1-ax \,.\, 1-ax^2} + \frac{ax^3(1-ax^4)^2}{1-ax \,.\, 1-ax^2 \,.\, 1-ax^3},$$

and in general

$$\mathrm{V}_p(x) = \frac{ax^{p-1}(1-ax^{p-1})^{p-3}}{1-ax \,.\, 1-ax^3 \ldots\ldots 1-ax^{p-2}} + \frac{ax^{p-1}(1-ax^{p-1})^{p-4}(1-ax^p)}{1-ax \,.\, 1-ax^2 \ldots . 1-ax^{p-2}}$$

$$+ \ldots\ldots + \frac{ax^{p-1}(1-ax^p)^{p-3}}{1-ax \,.\, 1-ax^2 \ldots . 1-ax^{p-2}} + \frac{ax^{p-1}(1-ax^p)^{p-2}}{1-ax \,.\, 1-ax^2 \ldots . 1-ax^{p-2} \,.\, 1-ax^{p-1}}.$$

The simplest cases, omitting the trivial one corresponding to $p = 1$, are

$$\frac{1}{1-x \,.\, 1-ax \,.\, 1-ax^2} = \frac{1}{(1-ax^2)^2}\left\{\frac{1}{1-x} + \frac{ax}{1-ax}\right\};$$

$$\frac{1}{1-x \,.\, 1-ax \,.\, 1-ax^2 \,.\, 1-ax^3} = \frac{1}{(1-ax^3)^3}\left\{\frac{1}{1-x} + \frac{ax}{1-ax} + \frac{ax^2}{1-ax} + \frac{ax^2(1-ax^3)}{1-ax \,.\, 1-ax^2}\right\};$$

$$\frac{1}{1-x.1-ax.1-ax^2.1-ax^3.1-ax^4} = \frac{1}{(1-ax^4)^4}\left\{\frac{1}{1-x} + \frac{ax}{1-ax} + \frac{ax^2}{1-ax} + \frac{ax^2(1-ax^3)}{1-ax.1-ax^2}\right.$$
$$\left. + \frac{ax^3(1-ax^3)}{1-ax.1-ax^2} + \frac{ax^3(1-ax^4)}{1-ax.1-ax^2} + \frac{ax^3(1-ax^4)^2}{1-ax.1-ax^2.1-ax^3}\right\}.$$

The fractions, in the brackets { }, may be united in batches, but I prefer to leave them as written, as the law of development is shown the better. Also the fraction $\frac{1}{1-ax^p}$ may be cancelled if desired. I have not done so, in order to keep in touch with the arithmetic.

Inspection of these expansions establishes the isolation therein independently.

Art. 26. In the function

$$\frac{1}{1-x.1-ax.1-ax^2....1-ax^p},$$

the coefficient of a^{q-1} is, as is well known,

$$\frac{1-x^q.1-x^{q+1}....1-x^{q+p-2}}{1-x.1-x^2....1-x^{p-1}}x^{q-1}.$$

Hence the coefficient of $a^{q-1}x^{p(q-1)}$ in the former is equal to the coefficient of $x^{p(q-1)}$ in the latter; *i.e.*, to the coefficient of $x^{(p-1)(q-1)}$ in

$$\frac{1-x^q.1-x^{q+1}....1-x^{q+p-2}}{(1-x)^2\,1-x^2....1-x^{p-1}},$$

which we know to be $\binom{p+q-2}{p-1}$, verifying our result.

For a given value of s the partitions are enumerated by the coefficient of $a^q b^s x^{pq}$ in the product

$$\frac{1}{1-x}\left(1+\frac{abx}{1-ax}\right)\left(1+\frac{abx^2}{1-ax^2}\right)...\left(\frac{abx^p}{1-ax^p}\right);$$

or, the same thing, by the coefficient of $(ax^p)^{q-1}$ in the function

$$\frac{a^{s-1}}{1-x.1-ax^p}\sum_{\mathrm{k}}\frac{x^{k_1+k_2+...+k_{s-1}}}{(1-ax^{k_1})(1-ax^{k_2})...(1-ax^{k_{s-1}})},$$

wherein $k_1, k_2, \ldots k_{s-1}$ denote any selection of $s-1$ different integers drawn from the natural series 1, 2, 3, . . . $p-1$.

This coefficient, from previous work, has the value

$$\binom{p-1}{s-1}\binom{q-1}{s-1}$$

hence the portion of this function which consists only of powers of ax^p is

$$\binom{p-1}{s-1}\left\{(ax^p)^{s-1} + \binom{s}{1}(ax^p)^s + \binom{s+1}{2}(ax^p)^{s+1} + \dots\right\}$$

or

$$\binom{p-1}{s-1}\frac{(ax^p)^{s+1}}{(1-ax^p)^s}.$$

Hence also from the function

$$\frac{a^{s-1}}{1-x}\Sigma\frac{x^{k_1+k_2+\dots+k_{s-1}}}{(1-ax^{k_1})(1-ax^{k_2})\dots(1-ax^{k_{s-1}})}$$

we can isolate the portion

$$\binom{p-1}{s-1}\left(\frac{ax^p}{1-ax^p}\right)^{s-1}.$$

Ex. gr. for $p = 3$, $s = 2$ we can verify that

$$2\frac{ax^3}{1-ax^3}$$

can be isolated from

$$\frac{ax}{1-x\,.\,1-ax} + \frac{ax^2}{1-x\,.\,1-ax^2}.$$

Art. 27. Before generalising the foregoing it will be proper to give another correspondence between the compositions and partitions of unipartite numbers which leads readily to theorems concerning the generating functions of partitions when the parts are *unrepeated.*

Writing down any composition of the unipartite p, viz.:

$$(p_1p_2p_3\,\dots\,,\,p_s),$$

we can at once construct a regularised partition, viz.:—

$$(p_1,\, p_1+p_2,\, p_1+p_2+p_3,\, \dots\,,\, p)$$

of the number $sp_1 + (s-1)p_2 + (s-2)p_3 + \dots + p_s$.

The correspondence is between the compositions of p into s parts and the partitions of all unipartite numbers into s *unequal* parts limited in magnitude to p and possessing a part p.

The numbers whose partitions appear are the natural series extending from

$$p + \binom{s}{2},\ \text{to}\ sp - \binom{s}{2}.$$

For the enumeration we must take the coefficient of $a^s x^{sp-\binom{s}{2}}$ in the development of the generating function

$$\frac{ax^p}{1-x}(1+ax)(1+ax^2)(1+ax^3)\ldots\ldots(1+ax^{p-1});$$

or the coefficient of $(ax^{p-\frac{1}{2}s})^{s-1}$ in

$$\frac{1}{1-x}(1+ax)(1+ax^2)(1+ax^3)\ldots(1+ax^{p-1}).$$

But the number of compositions of p into s parts is

$$\binom{p-1}{s-1},$$

and thence we see that a term in the development of

$$\frac{1}{1-x}(1+ax)(1+ax^2)(1+ax^3)\ldots(1+ax^{p-1}),$$

is

$$\binom{p-1}{s-1}a^{s-1}x^{(p-\frac{1}{2}s)(s-1)};$$

and, giving s successive values, we can isolate, from this product, a portion

$$1+\binom{p-1}{1}ax^{p-1}+\binom{p-1}{2}a^2x^{2p-3}+\binom{p-1}{3}a^3x^{3p-6}+\ldots+a^{p-1}x^{\binom{p}{2}};$$

or symbolically $(1+ax^p)^{p-1}$, where after expansion $x^{\mu p}$ is to be replaced by $x^{\mu p-\frac{1}{2}\mu(\mu+1)}$.

Art. 28. By leaving s unspecified we can readily reach a theorem concerning the product,

$$\frac{1}{1-x}(1+x)(1+x^2)(1+x^3)\ldots(1+x^{p-1}).$$

It is easy to show that the coefficient of $x^{\binom{p}{2}}$, in the development, is 2^{p-1}. We may say that the number of partitions of $\binom{p}{2}$ and all lower numbers into unrepeated parts not exceeding $p-1$ in magnitude is 2^{p-1}. For $p=5$ these partitions are:

4321				
432	43	31	4	
431	42	21	3	, 16 in number.
421	41		2	
321	32		1	
			0	

Art. 29. It is obvious that, by the same process, we can obtain a correspondence

between the compositions and partitions of multipartite numbers. In the bipartite case we pass from any composition

$$(\overline{p_1q_1}\ \overline{p_2q_2}\ \overline{p_3q_3} \cdot\ \cdot \overline{p_sq_s})$$

to the regularised partition

$$(\overline{p_1q_1}\ \overline{p_1 + p_2,\ q_1 + q_2,}\ \overline{p_1 + p_2 + p_3,\ q_1 + q_2 + q_3} \ldots \overline{pq})$$

of a certain bipartite number.

The correspondence is between the compositions of $\overline{pq}$ into s parts and the partitions of all bipartite numbers into s *unrepeated* biparts, the parts of the biparts being limited in magnitude to p and q respectively, and the highest bipart being $\overline{pq}$.

Or, we may strike out the highest bipart $\overline{pq}$, and then the partition is into $s - 1$ unrepeated biparts, the parts of the biparts being limited as before. The partitions are subject to the further restriction that they are regularised in the sense that the unipartite partitions of p and q, that appear in the bipartition, are separately regularised.

Art. 30. Instead of insisting upon this two-fold regularisation, we may, starting from the composition

$$(\overline{p_1q_1},\ \overline{p_2q_2},\ \overline{p_3q_3}, \ldots \overline{p_sq_s}),$$

proceed to the singly regularised partition

$$(\overline{p_1q_1}\ \overline{p_1 + p_2,\ q_2}\ \overline{p_1 + p_2 + p_3,\ q_3} \ldots \overline{pq_s}).$$

There are, in fact, various ways of forming connecting links between compositions and partitions of multipartite numbers whatever the order of multiplicity. These methods may be pursued at pleasure so as to obtain results of more or less interest.

§ 3.

Art. 31. The correspondence set forth between unipartite partitions and bipartite compositions naturally suggests the possibility of a similar correspondence between bipartite partitions and tripartite compositions, and generally between m-partite partitions and $m + 1$-partite compositions.

For the graph of the tripartite number $\overline{pqr}$, we take $r + 1$ similar graphs of the bipartite $\overline{pq}$, and place them similarly with corresponding lines parallel, and like points lying on straight lines; the graph is completed by drawing these straight lines, which are in a new direction, say the γ direction.

There are three directions through each point of the graph (see *loc. cit.*, Art. 31). There are $\binom{p+q+r}{p,\ q}$* lines of route along which the tripartite compositions are

* This notation explains $\frac{(p+q+r)!}{p!\ q!\ r!}$ and so in similar cases.

depicted, one line of route for each permutation of the symbols in the product $\alpha^p\beta^q\gamma^r$. A study of these permutations shows the connection with a certain class of bipartite partitions.

Consider a permutation

$$\alpha^{p_1}\beta^{q_1}\gamma^{r_1}\,\alpha^{p_2}\beta^{q_2}\gamma^{r_2}\,\alpha^{p_3}\beta^{q_3}\gamma^{r_3}\dots\alpha^{p_s}\beta^{q_s}\gamma^{r_s},$$

which is *not* the most general permutation, but such that, in regard to any section

$$\alpha^{p_k}\beta^{q_k}\gamma^{r_k}$$

of the permutation

(1.) r_k must be superior to zero except when $k = s$.
(2.) p_k, q_k may be either, but not both, zero, except when $k = 1$.

The permutation has $\gamma\alpha$ and $\gamma\beta$ contacts, but no $\beta\alpha$ contact.

In the reticulation corresponding thereto, we have lines of route with $\gamma\alpha$ and $\gamma\beta$ bends but *not* with $\beta\alpha$ bends. All the lines of route with $\beta\alpha$ bends are excluded from consideration. From the permutation we can form a bipartite partition.

$$(\overline{p_1q_1}^{r_1}\ \overline{p_1+p_2, q_1+q_2}^{r_2}\ \overline{p_1+p_2+p_3, q_1+q_2+q_3}^{r_3}\dots\overline{p_1+p_2+\dots+p_s, q_1+q_2+\dots+q_s}^{r_s}),$$

which is regularised in the sense that the partitions of the unipartites p, q, that appear are each separately regularised.

The two parts of the bipartite number thus partitioned are

$$r_1p_1 + r_2(p_1+p_2) + r_3(p_1+p_2+p_3) + \dots + r_s(p_1+p_2+\dots+p_s),$$
$$r_1q_1 + r_2(q_1+q_2) + r_3(q_1+q_2+q_3) + \dots + r_s(q_1+q_2+\dots+q_s).$$

The associated principal composition is

$$(\overline{p_1q_1r_1}\ \overline{p_2q_2r_2}\ \overline{p_3q_3r_3}\dots\overline{p_sq_sr_s}).$$

As before, consider the contacts r_1p_2, r_2p_3, &c. . . . Looking at the whole of the principal compositions, observe that a $\gamma\alpha$ contact in the permutation yields a contact r_kp_{k+1} in the composition in which r_k and p_{k+1} are both superior to zero, say a positive-positive contact. A $\gamma\beta$ contact yields a positive-zero contact and a $\beta\alpha$ contact a zero-positive contact. Hence the present correspondence is only concerned with compositions which possess positive-positive and positive-zero contacts, and not with those which involve contacts of other natures. The bipartite partitions are those of all bipartite numbers into biparts whose parts are limited to p and q respectively in magnitude and whose biparts are limited to r in number.

Art. 32. We have then a one-to-one correspondence. Each bipartite partition of the nature considered is represented graphically by a line of route in a tripartite reticulation. If we please we may regard a pair of bipartite partitions as represented by a line of route, for from the permutation we are also led to the complementary partition,

$$(\overline{p - p_1, q - q_1}^{r_1} \quad \overline{p - p_1 - p_2, q - q_1 - q_2}^{r_2} \ldots),$$

in which p_1, q_1 *may* be both zero.

Art. 33. It has been shown that the number of lines of route which possess

$$s_{21}\ \beta\alpha \text{ bends,}$$
$$s_{32}\ \gamma\beta \quad ,, \quad ,$$
$$s_{31}\ \gamma\alpha \quad ,, \quad ,$$

is

$$\binom{s_{21} + s_{31}}{s_{21}} \binom{p}{s_{21} + s_{31}} \binom{q}{s_{32}} \binom{q + s_{31}}{s_{21} + s_{31}} \binom{r}{s_{32} + s_{31}};*$$

and that this number is the coefficient of

$$\lambda_{21}^{s_{21}} \lambda_{32}^{s_{32}} \lambda_{31}^{s_{31}} \alpha^p \beta^q \gamma^r$$

in the development of

$$(\alpha + \lambda_{21}\beta + \lambda_{31}\gamma)^p (\alpha + \beta + \lambda_{32})^q (\alpha + \beta + \gamma)^r.$$

Here $s_{21} = 0$, and the number in question becomes

$$\binom{p}{s_{31}} \binom{q}{s_{32}} \binom{q + s_{31}}{s_{31}} \binom{r}{s_{32} + s_{31}},$$

whilst the generating function becomes

$$(\alpha + \lambda_{31}\gamma)^p (\alpha + \beta + \lambda_{32}\gamma)^q (\alpha + \beta + \gamma)^r.$$

In this the coefficient of

$$\lambda_{32}^{s_{32}} \lambda_{31}^{s_{31}} \alpha^p \beta^q \gamma^r$$

is equal to the coefficient of the same term in the expansion of the fraction

$$\frac{1}{1 - \alpha - \beta - \gamma + \alpha\beta + (1 - \lambda_{32})\beta\gamma + (1 - \lambda_{31})\alpha\gamma - (1 - \lambda_{32})\alpha\beta\gamma},$$

which is

$$\frac{1}{(1 - \alpha)(1 - \beta)(1 - \gamma) - \lambda_{31}\alpha\gamma - \lambda_{32}\beta\gamma(1 - \alpha)},$$

and the verification is readily carried out.

* 'Phil. Trans.,' vol. 184 (*loc. cit.*), Arts. 34, *et seq.*

Art. 34. The number of lines of route which possess exactly s_{31} $\gamma\alpha$ bends and no $\beta\alpha$ bends is

$$\binom{p}{s_{31}}\binom{q+s_{31}}{s_{31}}\sum_{s_{32}}\binom{q}{s_{32}}\binom{r}{s_{32}+s_{31}}$$

or

$$\binom{p}{s_{31}}\binom{q+s_{31}}{s_{31}}\binom{q+r}{q+s_{31}} \text{ or } \binom{p}{s_{31}}\binom{r}{s_{31}}\binom{q+r}{r}.$$

Also, by putting $\lambda_{31} = \lambda_{32} = \lambda$ in the generating function, we find that the number of lines of route which have s $\gamma\beta$ and $\gamma\alpha$ bends but no $\beta\alpha$ bend is given by the coefficient of $\alpha^p\beta^q\gamma^r$ in

$$\frac{(\alpha\gamma+\beta\gamma-\alpha\beta\gamma)^s}{(1-\alpha)^{s+1}(1-\beta)^{s+1}(1-\gamma)^{s+1}}.$$

Art. 35. It will be convenient to give the complete correspondence in the case of some simple tripartite number, say, $\overline{222}$.

S_{31}	S_{32}	Permutation.	Composition.	Partition.	Number partitioned.
0	0	$\alpha^2\beta^2\gamma^2$	$(\overline{222})$	$(\overline{22}^2)$	$\overline{44}$
1	0	$\alpha\beta^2\gamma\alpha\gamma$	$(\overline{121}\ \overline{101})$	$(\overline{12}\ \overline{22})$	$\overline{34}$
1	0	$\alpha\beta^2\gamma^2\alpha$	$(\overline{122}\ \overline{100})$	$(\overline{12}^2)$	$\overline{24}$
1	0	$\beta^2\gamma\alpha^2\gamma$	$(\overline{021}\ \overline{201})$	$(\overline{02}\ \overline{22})$	$\overline{24}$
1	0	$\alpha\beta\gamma\alpha\beta\gamma$	$(\overline{111}\ \overline{111})$	$(\overline{11}\ \overline{22})$	$\overline{33}$
1	0	$\alpha\gamma\alpha\beta^2\gamma$	$(\overline{101}\ \overline{121})$	$(\overline{10}\ \overline{22})$	$\overline{32}$
1	0	$\beta\gamma\alpha^2\beta\gamma$	$(\overline{011}\ \overline{211})$	$(\overline{01}\ \overline{22})$	$\overline{23}$
1	0	$\beta^2\gamma^2\alpha^2$	$(\overline{022}\ \overline{200})$	$(\overline{02}^2)$	$\overline{04}$
1	0	$\alpha\beta\gamma^2\alpha\beta$	$(\overline{112}\ \overline{110})$	$(\overline{11}^2)$	$\overline{22}$
1	0	$\gamma\alpha^2\beta^2\gamma$	$(\overline{001}\ \overline{221})$	$(\overline{00}\ \overline{22})$	$\overline{22}$
1	0	$\alpha\gamma^2\alpha\beta^2$	$(\overline{102}\ \overline{120})$	$(\overline{10}^2)$	$\overline{20}$
1	0	$\beta\gamma^2\alpha^2\beta$	$(\overline{012}\ \overline{210})$	$(\overline{01}^2)$	$\overline{02}$
1	0	$\gamma^2\alpha^2\beta^2$	$(\overline{002}\ \overline{220})$	$(\overline{00}^2)$	$\overline{00}$
0	1	$\alpha^2\beta\gamma\beta\gamma$	$(\overline{211}\ \overline{011})$	$(\overline{21}\ \overline{22})$	$\overline{43}$
0	1	$\alpha^2\beta\gamma^2\beta$	$(\overline{212}\ \overline{010})$	$(\overline{21}^2)$	$\overline{42}$
0	1	$\alpha^2\gamma\beta^2\gamma$	$(\overline{201}\ \overline{021})$	$(\overline{20}\ \overline{22})$	$\overline{42}$
0	1	$\alpha^2\gamma^2\beta^2$	$(\overline{202}\ \overline{020})$	$(\overline{20}^2)$	$\overline{40}$
1	1	$\alpha\beta\gamma\alpha\gamma\beta$	$(\overline{111}\ \overline{101}\ \overline{010})$	$(\overline{11}\ \overline{21})$	$\overline{32}$
1	1	$\alpha\beta\gamma\beta\gamma\alpha$	$(\overline{111}\ \overline{011}\ \overline{100})$	$(\overline{11}\ \overline{12})$	$\overline{23}$

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S_{31}	S_{32}	Permutation.	Composition.	Partition.	Number partitioned.
1	1	$\alpha\gamma\alpha\beta\gamma\beta$	$(\overline{101}\ \overline{111}\ \overline{010})$	$(\overline{10}\ \overline{21})$	$\overline{31}$
1	1	$\beta\gamma\alpha^2\gamma\beta$	$(\overline{011}\ \overline{201}\ \overline{010})$	$(\overline{01}\ \overline{21})$	$\overline{22}$
1	1	$\alpha\gamma\beta^2\gamma\alpha$	$(\overline{101}\ \overline{021}\ \overline{100})$	$(\overline{10}\ \overline{12})$	$\overline{22}$
1	1	$\alpha\gamma\alpha\gamma\beta^2$	$(\overline{101}\ \overline{101}\ \overline{020})$	$(\overline{10}\ \overline{20})$	$\overline{30}$
1	1	$\beta\gamma\beta\gamma\alpha^2$	$(\overline{011}\ \overline{011}\ \overline{200})$	$(\overline{01}\ \overline{02})$	$\overline{03}$
1	1	$\gamma\alpha^2\beta\gamma\beta$	$(\overline{001}\ \overline{211}\ \overline{010})$	$(\overline{00}\ \overline{21})$	$\overline{21}$
1	1	$\alpha\gamma\beta\gamma\alpha\beta$	$(\overline{101}\ \overline{011}\ \overline{110})$	$(\overline{10}\ \overline{11})$	$\overline{21}$
1	1	$\gamma\alpha^2\gamma\beta^2$	$(\overline{001}\ \overline{201}\ \overline{020})$	$(\overline{00}\ \overline{20})$	$\overline{20}$
1	1	$\gamma\beta^2\gamma\alpha^2$	$(\overline{001}\ \overline{021}\ \overline{200})$	$(\overline{00}\ \overline{02})$	$\overline{02}$
1	1	$\gamma\beta\gamma\alpha^2\beta$	$(\overline{001}\ \overline{011}\ \overline{210})$	$(\overline{00}\ \overline{01})$	$\overline{01}$
2	0	$\beta^2\gamma\alpha\gamma\alpha$	$(\overline{021}\ \overline{101}\ \overline{100})$	$(\overline{02}\ \overline{12})$	$\overline{14}$
2	0	$\beta\gamma\alpha\beta\gamma\alpha$	$(\overline{011}\ \overline{111}\ \overline{100})$	$(\overline{01}\ \overline{12})$	$\overline{13}$
2	0	$\gamma\alpha\beta^2\gamma\alpha$	$(\overline{001}\ \overline{121}\ \overline{100})$	$(\overline{00}\ \overline{12})$	$\overline{12}$
2	0	$\beta\gamma\alpha\gamma\alpha\beta$	$(\overline{011}\ \overline{101}\ \overline{110})$	$(\overline{01}\ \overline{11})$	$\overline{12}$
2	0	$\gamma\alpha\beta\gamma\alpha\beta$	$(\overline{001}\ \overline{111}\ \overline{110})$	$(\overline{00}\ \overline{11})$	$\overline{11}$
2	0	$\gamma\alpha\gamma\alpha\beta^2$	$(\overline{001}\ \overline{101}\ \overline{120})$	$(\overline{00}\ \overline{10})$	$\overline{10}$
0	2	$\alpha^2\gamma\beta\gamma\beta$	$(\overline{201}\ \overline{011}\ \overline{010})$	$(\overline{20}\ \overline{21})$	$\overline{41}$

There are 36 partitions.

The first two columns show the nature of the permutation in regard to $\gamma\alpha$ and $\gamma\beta$ contacts and the nature of the composition in regard to positive-positive and positive-zero contacts. The partitions are into two parts, zero not excluded, and have regard to bipartite numbers extending from $\overline{44}$ to $\overline{00}$. They are doubly regularised by ascending magnitude, and the figures of the parts do not exceed 2, 2 the first two figures of the tripartite.

If we write down the partitions of 4 into two parts, zeros not excluded, limited not to exceed 2 in magnitude, viz. :—

$$22,\ 12,\ 02,\ 11,\ 01,\ 00,$$

the ascending order of part magnitude being adhered to, we can obtain one of the 36 partitions by combining any one of these partitions with itself or any other of the 6.

Thus the fourth of the above partitions is obtained by combining the unipartite partitions

$$02, \quad 22,$$

and from any two unipartite partitions

$$ab, \quad cd,$$

we proceed to the bipartite partition

$$(\overline{ac} \quad \overline{bd}).$$

The number is thus shown to be $6 \times 6 = 36$.

Art. 36. In general, when the tripartite is $\overline{pqr}$, the partitions are into r parts, zeros not excluded, the first and second figures of the biparts being limited to p and q respectively.

The bipartite numbers partitioned extend from

$$\overline{p \times r, q \times r} \quad \text{to} \quad \overline{00}.$$

The partitions are doubly regularised and may be enumerated by observing that we have to combine every partition of $p \times r$ and lower unipartite numbers into r parts, zeros not excluded, and no part exceeding p in magnitude, with every partition of $q \times r$ and lower numbers into r parts, zeros not excluded, and no part exceeding q in magnitude.

Hence (see *ante*, Art. 12) the number of partitions is

$$\binom{p+r}{r}\binom{q+r}{r}.$$

This expression also enumerates (1) the compositions which have only positive-positive and positive-zero contacts; (2) the lines of route in the tripartite reticulation which are without $\beta\alpha$ bends; (3) the permutations of $\alpha^p\beta^q\gamma^r$ which are without $\beta\alpha$ contacts.

Art. 37. The truth of the theorem may be seen also as follows :—Suppose a solid reticulation and take the directions α, β, γ as axes of x, y, and z meeting at the origin of the lines of route. The face of the solid in the plane xz is a *bipartite* reticulation in which $\binom{p+r}{r}$ lines of route may be drawn; similarly $\binom{q+r}{r}$ lines of route may be drawn in the bipartite reticulation which lies in the plane yz. One of the former lines of route is an orthogonal projection of a tripartite line of route on the plane xz; one of the latter is an orthogonal projection on the plane yz; any one of the former may be associated with any one of the latter, and such a pair uniquely determines a tripartite line of route which does not possess $\beta\alpha$ bends. This may be clearly seen by considering the permutation

$$\alpha^{p_1}\beta^{q_1}\gamma^{r_1}\ \alpha^{p_2}\beta^{q_2}\gamma^{r_2}\ldots;$$

suppression alternately of the letters β and α yields two permutations, viz. :—

$$\alpha^{p_1}\gamma^{r_1}\alpha^{p_2}\gamma^{r_2}\ldots$$
$$\beta^{q_1}\gamma^{r_1}\beta^{q_2}\gamma^{r_2}\ldots$$

which express the bipartite lines of route which are the projections on the planes xz, yz respectively. Since the tripartite permutation involves no $\beta\alpha$ contacts, we see that these two permutations uniquely determine the permutation

$$\alpha^{p_1}\beta^{q_1}\gamma^{r_1}\alpha^{p_2}\beta^{q_2}\gamma^{r_2}\ldots$$

Hence the number of lines of route in question is

$$\binom{p+r}{r}\binom{q+r}{r}.$$

Art. 38. Hence also the interesting summation formula

$$\sum_{s_{31}}\sum_{s_{32}}\binom{p}{s_{31}}\binom{q}{s_{32}}\binom{q+s_{31}}{s_{31}}\binom{r}{s_{31}+s_{32}}=\binom{p+r}{r}\binom{q+r}{r}.$$

Observe that the expression further enumerates the lines of route with r, $\beta\alpha$ bends in the reticulation of the bipartite $\overline{p+r, q+r}$.

A generating function which enumerates these partitions is

$$\frac{1}{1-x\,.\,1-a\,.\,1-ax\ldots1-ax^p\,1-y\,.\,1-b\,.\,1-by\ldots1-by^q},$$

in which the coefficient of $(abx^py^q)^r$ must be sought.

The compositions that appear are the principal ones along lines of route which have no $\beta\alpha$ bends. We may strike out the last part of the composition whenever its last figure is zero, and then the compositions are not of the single tripartite $\overline{222}$, but of the 9 tripartites extending from $\overline{222}$ to $\overline{002}$, the last figure being 2, and the first two figures not exceeding 2, 2 respectively. The compositions are into 2, or fewer parts. Generally the compositions appear of the $(p+1)(q+1)$ tripartites extending from $\overline{pqr}$ to $\overline{00r}$, the last figure being r, and the first two figures not exceeding p, q, respectively. The compositions are into r, or fewer parts, no part having the last figure zero.

The partitions present themselves in complementary pairs. To every partition $(\overline{ab}\;\overline{cd}\ldots)$ corresponds another $(\overline{p-a, q-b},\ \overline{p-c, q-d}\ldots)$ the numbers partitioned being respectively $\overline{a+c+\ldots, b+d+\ldots}$ and $\overline{rp-a-c-\ldots, rq-b-d-\ldots}$ *Ex. gr.*, the complementary partitions $(\overline{02}\;\overline{22})$, $(\overline{20}\;\overline{00})$ of the bipartites $\overline{24}$, $\overline{20}$ Certain partitions are self-complementary. The number partitioned is then $\overline{\tfrac{1}{2}rp, \tfrac{1}{2}rq}$.

Art. 39. We may enumerate the partitions which, excluding zero, involve k different parts. Let s_{23}, s_{13}, represent the number of $\beta\gamma$ and $\alpha\gamma$ contracts in a permutation. If $s_{23} + s_{13} = k$, the corresponding partition possesses k different parts other than zero. The lines of route are such as have no $\beta\alpha$ bends, s_{13} $\alpha\gamma$ bends and s_{23} $\beta\gamma$ bends. Reversing the permutation we have a similar number of lines of route which have no $\alpha\beta$ bends, s_{13} $\gamma\alpha$ bends, and s_{23} $\gamma\alpha$ bends. Now interchange α and β and replace the reticulation of the tripartite $\overline{pqr}$ by that of $\overline{qpr}$. In this new reticulation we have the same number of lines of route which have no $\beta\alpha$ bends, s_{13} $\gamma\beta$ bends, and s_{23} $\gamma\alpha$ bends. This number has been shown to be

$$\binom{q}{s_{23}}\binom{p+s_{23}}{s_{23}}\binom{p}{s_{13}}\binom{r}{s_{13}+s_{23}}.$$

Art. 40. Hence the bipartite partitions possessing k different parts other than zero are enumerated by

$$\binom{r}{k}\sum_{s_{23}}\binom{q}{s_{23}}\binom{p+s_{23}}{s_{23}}\binom{p}{k-s_{23}}.$$

Theorem.—Having under consideration the doubly-regularised partitions of all bipartite numbers into r parts, zero parts included, such that the figures of the parts are limited in magnitude to p and q respectively, the number of partitions which possess exactly k different parts, other than zero, is

$$\binom{r}{k}\sum_{s_{23}}\binom{q}{s_{23}}\binom{p+s_{23}}{s_{23}}\binom{p}{k-s_{23}},$$

s_{23} assuming all compatible values.

This result may be verified in the case of the tripartite $\overline{222}$ from the table given above. As an additional verification, consider the tripartite $\overline{123}$. For $k = 2$, we have

Permutations.	Compositions.	Partitions.
$\gamma\alpha\beta\gamma\beta\gamma$	$(\overline{001}\ \overline{111}\ \overline{011})$	$(\overline{00}\ \overline{11}\ \overline{12})$
$\gamma\alpha\gamma\beta^2\gamma$	$(\overline{001}\ \overline{101}\ \overline{021})$	$(\overline{00}\ \overline{10}\ \overline{12})$
$\gamma\alpha\gamma\beta\gamma\beta$	$(\overline{001}\ \overline{101}\ \overline{011})$	$(\overline{00}\ \overline{10}\ \overline{11})$
$\alpha\beta\gamma^2\beta\gamma$	$(\overline{112}\ \overline{011})$	$(\overline{11}^2\ \overline{12})$
$\alpha\beta\gamma\beta\gamma^2$	$(\overline{111}\ \overline{012})$	$(\overline{11}\ \overline{12}^2)$
$\alpha\gamma^2\beta^2\gamma$	$(\overline{102}\ \overline{021})$	$(\overline{10}^2\ \overline{12})$
$\alpha\gamma^2\beta\gamma\beta$	$(\overline{102}\ \overline{011})$	$(\overline{10}^2\ \overline{11})$
$\alpha\gamma\beta\gamma^2\beta$	$(\overline{101}\ \overline{012})$	$(\overline{10}\ \overline{11}^2)$
$\alpha\gamma\beta^2\gamma^2$	$(\overline{101}\ \overline{022})$	$(\overline{10}\ \overline{12}^2)$

Permutations.	Compositions.	Partitions.
$\gamma\beta\gamma\beta\gamma\alpha$	$(\overline{001}\ \overline{011}\ \overline{011})$	$(\overline{00}\ \overline{01}\ \overline{02})$
$\gamma\beta\gamma\alpha\beta\gamma$	$(\overline{001}\ \overline{011}\ \overline{111})$	$(\overline{00}\ \overline{01}\ \overline{12})$
$\beta^2\gamma^2\alpha\gamma$	$(\overline{022}\ \overline{101})$	$(\overline{02}^2\ \overline{12})$
$\beta^2\gamma\alpha\gamma^2$	$(\overline{021}\ \overline{102})$	$(\overline{02}\ \overline{12}^2)$
$\beta\gamma^2\alpha\beta\gamma$	$(\overline{012}\ \overline{111})$	$(\overline{01}^2\ \overline{12})$
$\beta\gamma^2\alpha\gamma\beta$	$(\overline{012}\ \overline{101})$	$(\overline{01}^2\ \overline{11})$
$\beta\gamma^2\beta\gamma\alpha$	$(\overline{012}\ \overline{011})$	$(\overline{01}^2\ \overline{02})$
$\beta\gamma\alpha\beta\gamma^2$	$(\overline{011}\ \overline{112})$	$(\overline{01}\ \overline{12}^2)$
$\beta\gamma\alpha\gamma^2\beta$	$(\overline{011}\ \overline{102})$	$(\overline{01}\ \overline{11}^2)$
$\beta\gamma\beta\gamma^2\alpha$	$(\overline{011}\ \overline{012})$	$(\overline{01}\ \overline{02}^2)$
$\gamma\beta^2\gamma\alpha\gamma$	$(\overline{001}\ \overline{021}\ \overline{101})$	$(\overline{00}\ \overline{02}\ \overline{12})$
$\gamma\beta\gamma\alpha\gamma\beta$	$(\overline{001}\ \overline{011}\ \overline{101})$	$(\overline{00}\ \overline{01}\ \overline{11})$

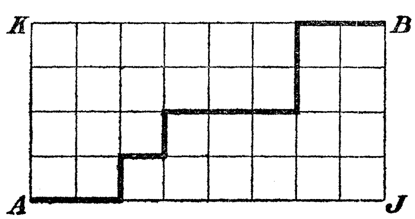
21 partitions; while for the enumeration, giving s_{23} the values 0, 1, 2 in succession with $k = 2$, $p = 1$, $q = 2$, $r = 3$.

$$\binom{3}{2}\left\{\binom{2}{0}\binom{1+0}{0}\binom{1}{2}+\binom{2}{1}\binom{1+1}{1}\binom{1}{1}+\binom{2}{2}\binom{1+2}{2}\binom{1}{0}\right\}$$
$$= 3\,(1 \times 1 \times 0 + 2 \times 2 \times 1 + 1 \times 3 \times 1) = 3 \times 7 = 21.$$

The foregoing particular theory of the correspondence that exists between tripartite compositions and bipartite partitions is, for present purposes, sufficiently indicative of the general correspondence between $(m + 1)$-partite compositions and a certain regularised class of m-partite partitions.

§ 4. Constructive Theory.

Art. 41. Given a line of route in a bipartite reticulation it may be necessary to enumerate the lines of route which lie altogether on either side of it.



Thus in respect of the line of route delineated in the reticulation AB, lines of route exist which, throughout their entire course, are either coincident with it, or lie on the side of it towards J. Such lines of route may be termed inferior or subjacent to the given line of route. Similarly those lines of route which, everywhere, are either coincident with the given line, or on the side remote from J, may be termed superior or superjacent lines of route in respect of the given line. All lines are thus accounted for with the exception of those which cross the given line passing from the side towards J to the side remote from J, or *vice versâ*; these may be termed transverse lines in respect of the given line.

Art. 42. I am concerned, at present, with those lines which are subjacent to a given line, though it will be remarked that the superjacent and transverse lines also suggest questions of interest. A given line of route defines a bipartite principal composition

$$(\overline{p_1 q_1}\ \overline{p_2 q_2}\ \ldots),$$

and a unipartite south-easterly partition

$$(\overline{p - p_1}^{q_1}\ \overline{p - p_1 - p_2}^{q_2}\ \ldots).$$

The bipartite compositions and the unipartite partitions, defined by the subjacent lines of route, are termed subjacent to the given composition and the given partition respectively.

We may draw a number of lines of route, each of which is subjacent to the given line and not transverse to any other of the number. We thus obtain what may be termed a subjacent succession of lines giving rise to a subjacent succession of unipartite partitions.

These regularised partitions may be

$$(a_1 a_2 a_3 \ldots),\quad (b_1 b_2 b_3 \ldots),\quad (c_1 c_2 c_3 \ldots) \ldots .$$

and they are such that the partitions

$$(a_1 b_1 c_1 \ldots),\ (a_2 b_2 c_2 \ldots),\ (a_3 b_3 c_3 \ldots) .$$

are also regularised.

It is clear also that the subjacent succession of lines represents the multipartite partition

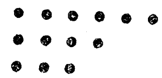
$$(\overline{a_1 a_2 a_3 \ldots},\ \overline{b_1 b_2 b_3 \ldots},\ \overline{c_1 c_2 c_3 \ldots},\ \overline{\ldots\ldots})$$

of the multipartite numbers

$$(\overline{a_1 + b_1 + c_1 + \ldots},\ \overline{a_2 + b_2 + c_2 + \ldots},\ \overline{a_3 + b_3 + c_3 + \ldots},\ \overline{\ldots}).$$

This partition may be termed "graphically regularised" by reason of its origination in a subjacent succession of lines in the bipartite graph. This species of regularisation is the natural extension to three dimensions of SYLVESTER's graphical method in two dimensions.

Art. 43. SYLVESTER represents the partition $(\alpha_1\alpha_2\alpha_3\ldots)$ of a unipartite number A by the graph



the lines containing $\alpha_1, \alpha_2, \alpha_3\ldots$ nodes successively.

The same graph also represents a multipartite number $(\overline{\alpha_1\alpha_2\alpha_3\ldots})$ whose content is A, viz.,

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots = \text{A}.$$

SYLVESTER's theory is, in fact, not only a theory of the partitions of a number A, but also a theory of the multipartite numbers whose content is A. For purpose of generalization I prefer to regard it from the latter point of view.

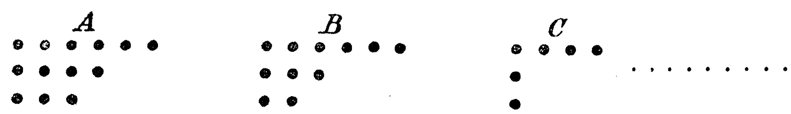
If we consider the graphically regularised partition

$$(\overline{a_1a_2a_3\ldots}, \overline{b_1b_2b_3\ldots}, \overline{c_1c_2c_3\ldots}, \overline{\ldots\ldots})$$

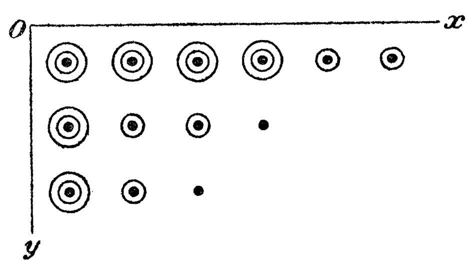
of the multipartite number

$$(\overline{a_1 + b_1 + c_1 + \ldots}, \overline{a_2 + b_2 + c_2 + \ldots}, \overline{a_3 + b_3 + c_3 + \ldots}, \overline{\ldots\ldots})$$

and write down the Sylvester-graphs of the multipartite numbers which are the parts of the partition



it is clear that we may pile B upon A, and then C upon B, &c., and thus form a three-dimensional graph of the partition



which is regularised in three-dimensions just as the Sylvester-graphs are regularised in two.

This representation is only possible when the subjacent succession of lines is insisted upon.

Art. 44. Every Sylvester-graph in two dimensions is representative of two unipartite partitions; it may, in fact, be read by lines or by columns, and when the two readings are identical the graph is said to be self conjugate.

In this enlarged theory every graph denotes 3! graphically-regularised multipartite partitions; of the same total content, but not, as a rule, appertaining to the same multipartite number.

Take coordinate axes as shown, the axis of z being perpendicular to the plane of the paper. We read as follows:—

Planes parallel to the plane xy and in direction Ox

$$(\overline{643}\ \overline{632}\ \overline{411}).$$

Planes parallel to plane xy and in direction Oy

$$(\overline{333211}\ \overline{332111}\ \overline{311100}).$$

Planes parallel to plane yz and in direction Oy

$$(\overline{333}\ \overline{331}\ \overline{321}\ \overline{211}\ \overline{110}\ \overline{110}).$$

Planes parallel to plane yz and in direction Oz

$$(\overline{333}\ \overline{322}\ \overline{321}\ \overline{310}\ \overline{200}\ \overline{200}).$$

Planes parallel to plane zx and in direction Oz

$$(\overline{333322}\ \overline{322100}\ \overline{321000}).$$

Planes parallel to plane zx and in direction Ox

$$(\overline{664}\ \overline{431}\ \overline{321}),$$

the multipartite numbers, of which these are partitions, being

$(\overline{16, 8, 6})$,

$(\overline{976422})$, content

$(\overline{13, 11, 6})$, 30

$(\overline{16, 8, 6})$,

$(\overline{976422})$,

$(\overline{13, 11, 6})$.

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The graph is therefore representative of three multipartite numbers and of two partitions of each.

Art. 45. A multipartite number has two characteristics. It may be r-partite, *i.e.*, it may consist of r figures, and its highest figure may be p. A multipartite partition has three characteristics. Each part may be r-partite; the highest figure may be p; the number of parts may be q. If the graph be formed of a multipartite partition with characteristics

$$r, p, q,$$

the five other readings yield partitions with characteristics :—

$$\begin{array}{ccc} p, & r, & q \\ q, & r, & p \\ r, & q, & p \\ p, & q, & r \\ q, & p, & r. \end{array}$$

The six partitions correspond to the six permutations of the three symbols p, q, r.

The two partitions which are r-partite appertain to the same multipartite number; similarly for the pairs which are p-partite and q-partite respectively. Hence the three multipartite numbers involved correspond to the three pairs of permutations so formed that in any pair the commencing symbol of each permutation is the same.

Art. 46. The consideration of graphs formed with a *given number* of nodes now leads to the theorem: "The enumeration of the graphically regularised r-partite partitions, into q parts and having p for the highest figure, gives the same number for each of the six ways in which the numbers p, q, r may be permuted."

Also the theorem :—

"The enumeration of the graphically regularised partitions which are at most r-partite, into q or fewer parts, the highest figure not exceeding p, gives the same number for each of the six ways in which the numbers p, q, r may be permuted."

The first theorem is concerned with fixed values of p, q, and r; the second with restricted values of these numbers. It is also clear that we may fix one or two of the numbers and leave the remaining two or one restricted.

Observe that this six-fold conjugation obtains even though equalities exist between the numbers p, q, r; they must be regarded always as different numbers. Sometimes, as we shall see, the correspondence is less than six-fold, but this does not depend solely upon the assignment of the numbers p, q, r.

If we regard the multipartite number appertaining to a partition and not merely the total content, we find that the partitions occur in pairs.

Quâ a given multipartite number, a partition which has q parts and a highest figure p is in association with one which has p parts and a highest figure q.

Thus of the multipartite number $(\overline{13.11.6})$ we have the partitions

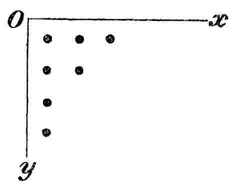
$$(\overline{333}\ \overline{331}\ \overline{321}\ \overline{211}\ \overline{110}\ \overline{110})$$

$$(\overline{664}\ \overline{431}\ \overline{321})$$

derived from the above written graph.

Art. 47. It is interesting to view the two-dimensional Sylvester-graphs from the three-dimensional standpoint.

Consider the graph



which, following SYLVESTER, denotes the unipartite partition (3211) of the unipartite number 7.

In this paper, the graph, read Sylvester-wise in the plane xy and in direction Ox, denotes the multipartite number $(\overline{3211})$ of content 7. SYLVESTER'S conjugate reading, plane xy and direction Oy, gives the partition (421) but here denotes the multipartite number $(\overline{421})$. There are four other readings in this theory. The six readings are

Plane xy	Direction	Ox	$(\overline{3211})$	$(p, q, r) = (3, 1, 4)$
,, xy	,,	Oy	$(\overline{421})$	$(p, q, r) = (4, 1, 3)$
,, yz	,,	Oy	(421)	$(p, q, r) = (4, 3, 1)$
,, yz	,,	Oz	$(\overline{1111}\ \overline{1100}\ \overline{1000})$	$(p, q, r) = (1, 3, 4)$
,, zx	,,	Oz	(3211)	$(p, q, r) = (3, 4, 1)$
,, zx	,,	Ox	$(\overline{111}\ \overline{110}\ \overline{100}\ \overline{100})$	$(p, q, r) = (1, 4, 3)$.

The three multipartite numbers

$$(7),\ \overline{421},\ (\overline{3211})$$

appear each in two partitions.

In general we establish, in regard to Sylvester-graphs, the six-fold correspondence between

(1) r-partite partitions, containing 1 part and a highest figure p.

(2) p-partite partitions, containing 1 part and a highest figure r.

(3) unipartite partitions, containing p parts and a highest figure r.
(4) r-partite partitions, containing p parts and a highest figure 1.
(5) unipartite partitions, containing r parts and a highest figure p.
(6) p-partite partitions, containing r parts and a highest figure 1.

In this enunciation we may substitute for r or p, or for both, the phrases "not exceeding r," "not exceeding p."

Art. 48. For a given number of nodes, in the simplest cases, it will be suitable to view the graphs of the graphically regularised partitions.

Omitting the trivial case of a single node, we have

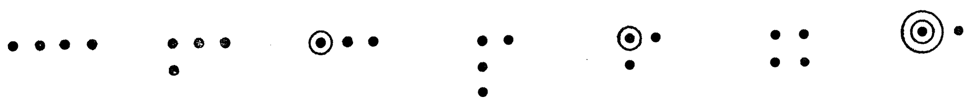
Number 2.

(2) $(\overline{1}\overline{1})$ (11)

Number 3.

(3) $(\overline{21})$ (21) $(\overline{111})$ $(\overline{11}\ \overline{10})$ (111)

Number 4.



(4) $(\overline{31})$ (31) $(\overline{211})$ $(\overline{21}\ \overline{10})$ $\overline{22}$ (211)

(22) $(\overline{1111})$ $(\overline{111}\ \overline{100})$ $(\overline{11}\ \overline{10}\ \overline{10})$ $(\overline{11}\ \overline{11})$ (1111)

The table is continued in an obvious manner. The *essentially* distinct graphs are for the

Number 2.

Number 3.

Number 4.

 *

and the whole of the partitions are obtainable by reading them in the various ways above explained.

Art. 49. It will be convenient to adopt in future another notation for the graphs; the number m will denote a vertical column of m nodes piled upon one another. The 13 graphs appertaining to the number 4 are written

1111	111	211	11	21	11	31
	1		1	1	11	
			1			
(4)	$(\overline{31})$	(31)	$(\overline{211})$	$(\overline{21}\ \overline{10})$	$(\overline{22})$	(211)

22	1	2	3	2	4
	1	1	1	2	
	1	1			
	1				
(22)	$(\overline{1111})$	$(\overline{111}\ \overline{100})$	$(\overline{11}\ \overline{10}\ \overline{10})$	$(\overline{11}\ \overline{11})$	(1111).

The essentially distinct graphs with the partitions appertaining to them are

1111	111	21	11
	1	1	11
(4)	$(\overline{31})$	$(\overline{21}\ \overline{10})$	$(\overline{22})$
$(\overline{1111})$	(31)		(22)
(1111)	$(\overline{211})$		$(\overline{11}\ \overline{11})$
	(211)		
	$(\overline{111}\ \overline{100})$		
	$(\overline{11}\ \overline{10}\ \overline{10})$		

Art. 50. Such graphs are either symmetrical, quasi-symmetrical, or unsymmetrical. The symmetrical graphs have three dimensional symmetry, and yield only one partition each.

The quasi-symmetrical have two-dimensional symmetry and yield three partitions each. The unsymmetrical yield each six partitions.

If F (x) be the enumerating generating function to a given content, we may write

$$F(x) = f_1(x) + 3f_2(x) + 6f_3(x),$$

* The interesting question arises as to the enumeration of the essentially distinct graphs of given content.

$f_1(x), f_2(x), f_3(x)$ being the generating functions for the essentially distinct graphs which are symmetrical, quasi-symmetrical, and unsymmetrical respectively.

Also we may write

$$F(x) = F_1(x) + F_2(x) + F_3(x),$$

where $F_1(x)$, $F_2(x)$, $F_3(x)$ are the generating functions of the partitions of the three natures.

The present theory is really the solidification of SYLVESTER's theory given in the 'American Journal of Mathematics' (*loc. cit.*). Already we have seen that the Sylvester-graphs are susceptible of a far wider interpretation than was at first anticipated. If we view these graphs from a two-dimensional standpoint, every graph is either symmetrical or unsymmetrical, the symmetrical class comprising all graphs which are self-conjugate. If, however, our standpoint be three-dimensional, there are no longer any symmetrical graphs. The two classes are the quasi-symmetrical and the unsymmetrical. A single exception to the above occurs where the graph is of unity. Moreover, the classes now do not comprise the same members. Certain graphs which were unsymmetrical from the first standpoint appear as quasi-symmetrical from the second.

Omitting the trivial symmetrical graph of unity every two-dimensional graph can be read either in three or six ways. The quasi-symmetrical class giving three readings, comprises the self-conjugate graphs and also those which consist of either a single line or a single column of nodes. The remaining graphs give six readings.

Ex. gr. The graph

$$11111$$

yields the three partitions (5), $(\overline{11111})$, (11111) being quasi-symmetrical from the three-dimensional standpoint although it is unsymmetrical in SYLVESTER's theory.

Also the self-conjugate graph

$$\begin{matrix}111\\1\\1\end{matrix}$$

yields the three partitions $(\overline{311})$, (311), $(\overline{111}\ \overline{100}\ \overline{100})$.

Such a graph as

$$\begin{matrix}111\\111\end{matrix}$$

being unsymmetrical in both theories yields six partitions

$$(\overline{33}),\ (33),\ (\overline{222}),\ (222),\ (\overline{111}\ \overline{111}),\ (\overline{11}\ \overline{11}\ \overline{11}).$$

Art. 51. The enumeration of the three-dimensional graphs that can be formed with

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a given number of nodes, corresponding to the regularised partitions of all multipartite numbers of given content, is a weighty problem. I have verified to a high order that the generating function of the complete system is

$$(1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}(1-x^4)^{-4}\ldots \textit{ad inf.},$$

and, so far as my investigations have proceeded, everything tends to confirm the truth of this conjecture.

I observe that, to negative signs *près*, the exponents are

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \ldots$$

viz., the figurate numbers of order 2.

The generating function which enumerates the two-dimensional graphs, is

$$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}(1-x^4)^{-1}\ldots$$

where (notice) the exponents are

$$1, \quad 1, \quad 1, \quad 1, \quad 1, \ldots$$

the figurate numbers of order 1.

Proceeding further back, we find that one-dimensional graphs are enumerated by

$$(1-x)^{-1}(1-x^2)^{0}(1-x^3)^{0}(1-x^4)^{0}\ldots$$

the numbers

$$1, \quad 0, \quad 0, \quad 0, \quad 0, \ldots$$

being the figurate numbers of order zero. Going forward again it is easy to verify up to a certain point that four-dimensional graphs (which it is quite easy to graphically realise in two dimensions) are enumerated by

$$(1-x)^{-1}\ (1-x^2)^{-3}\ (1-x^3)^{-6}\ldots,$$

where the exponents involve the figurate numbers of order 3.

The law of enumeration appears, conjecturally, to involve the successive series of figurate numbers.

Art. 52. Before proceeding to establish certain results, it may be proper, as illustrating the method pursued in this difficult investigation, to give other results which, at first mere conjectures, are gradually having the mark of truth stamped upon them.

Consider graphs in which only the numbers 1 and 2 appear. These are two-layer partitions. The enumeration to a high order is given by the generating function

$$(2\,;\,\infty\,;\,\infty) = (1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-2}(1-x^4)^{-2}\ldots$$

where the notation $(l ; m ; n)$ is employed to represent the generating function of partitions whose graphs are limited in height, breadth, and length by l, m, n respectively.

Similarly we shall find :—

$$(3 ; \infty ; \infty) = (1-x)^{-1}(1-x^2)^{-2}[(1-x^3)(1-x^4)\ldots]^{-3},$$

$$(4 ; \infty ; \infty) = (1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}[(1-x^4)(1-x^5)\ldots]^{-4},$$

$$(l ; \infty ; \infty) = (1-x)^{-1}(1-x^2)^{-2}\ldots(1-x^{l-1})^{-(l-1)}[(1-x^l)(1-x^{l+1})\ldots]^{-l},$$

$$(l ; 1 ; \infty) = (1-x)^{-1}(1-x^2)^{-1}\ldots(1-x^l)^{-1},$$

$$(l ; 2 ; \infty) = (1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-2}\ldots(1-x^l)^{-2}(1-x^{l+1})^{-1},$$

$$(l ; 3 ; \infty) = (1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}\ldots(1-x^l)^{-3}(1-x^{l+1})^{-2}(1-x^{l+2})^{-1},$$

$$(l ; \mathrm{m} ; \infty) = (1-x)^{-1}(1-x^2)^{-2}\ldots(1-x^{m-1})^{-(m-1)}\times[(1-x^m)\ldots(1-x^l)]^{-m},$$
$$\times(1-x^{l+1})^{-(m-1)}(1-x^{l+2})^{-(m-2)}\ldots(1-x^{l+m-1})^{-1},$$

if m be not greater than l ;

with an equivalent form

$$(l ; m ; \infty) = (1-x)^{-1}(1-x^2)^{-2}\ldots(1-x^{l-1})^{-l-1}\times[(1-x^l)\ldots(1-x^m)]^{-l}$$
$$\times(1-x^{m+1})^{-(l-1)}(1-x^{m+2})^{-(l-2)}\ldots(1-x^{l+m-1})^{-1},$$

if m be greater than l ;

and finally

$$(l ; m ; n) = \frac{1-x^{n+1}}{1-x}\cdot\frac{(1-x^{n+2})^2}{(1-x^2)^2}\ldots\frac{(1-x^{n+l-1})^{l-1}}{(1-x^{l-1})^{l-1}}$$
$$\times\left[\frac{1-x^{n+l}}{1-x^l}\cdot\frac{1-x^{n+l+1}}{1-x^{l+1}}\ldots\frac{1-x^{n+m}}{1-x^m}\right]^l$$
$$\times\frac{(1-x^{n+m+1})^{l-1}}{1-x^{m+1})^{l-1}}\cdot\frac{(1-x^{n+m+2})^{l-2}}{(1-x^{m+2})^{l-2}}\ldots\frac{1-x^{n+l+m-1}}{1-x^{l+m-1}},$$

a result which can be shown to be symmetrical in l, m and n, as ought, of course, to be the case.

This expression for $(l ; m ; n)$ can be exhibited in a more suggestive form, viz. :—

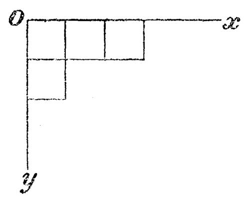
Writing $1 - x^s = (s)$

$$(l;\ m;\ n) = \frac{(n+1)(n+2)\ldots\ldots(l+m+n-1)}{(1)\quad(2)\quad\ldots\ldots\quad(l+m-1)}$$
$$\times\frac{(n+2)(n+3)\ldots\ldots(l+m+n-2)}{(2)\quad(3)\quad\ldots\ldots\quad(l+m-2)}\times\frac{(n+3)(n+4)\ldots\ldots(l+m+n-3)}{(3)\quad(4)\quad\ldots\ldots\quad(l+m-3)},$$

$\times$ . . . . to l factors or m factors, according as m or l is the greater.

Art. 53. In attempting to establish these results, it is easy to construct a generating function which contains implicitly the complete solution of the problems.

The problem itself may be enunciated in another manner which has points of great interest. A two-dimensional graph of SYLVESTER may be supposed formed by pushing



a number of cubes into a flat rectangular corner $y0x$ in such wise that the arrangement is immovable under the action of forces applied in the directions $x0$, $y0$.

It is clear that the number of such arrangements of n cubes is the number of two-dimensional graphs of n, or the number of partitions of the unipartite number n. Similarly, we may push a number of cubes into a three-dimensional rectangular corner, piling of cubes permissible, and such that the arrangement is immovable for forces applied in the three directions $x0$, $y0$, $z0$. The enumeration of these arrangements is the same as that in the problem under discussion.

Art. 54. First consider arrangements limited in the manner $(l\,;\ m\,;\ n) = (2\,;\ 1\,;\ \infty)$. We have such a graph as

$$\begin{array}{c} 2 \\ 2 \\ 2 \\ 1 \\ 1, \end{array}$$

obtained by writing a column of nodes, and over it another column of nodes, not exceeding the former in number.

We may take, as the generating function,

$$\frac{1}{(1-ax)(1-x/a)},$$

in which we are only concerned with that portion of the expansion which is integral as regards a. The function is, in fact, redundant since it involves terms which are superfluous, and we obtain the reduced or condensed generating function by putting a equal to unity in the portion we retain.

Since

$$\frac{1}{(1-ax)\left(1-\dfrac{x}{a}\right)} = \frac{1}{1-x^2}\left\{\frac{1}{1-ax} + \frac{\dfrac{x}{a}}{1-\dfrac{x}{a}}\right\}$$

the reduced generating function is

$$\frac{1}{(1-x)(1-x^2)},$$

and this is obviously correct, because from the form of the graph we have merely to enumerate the ways of partitioning numbers with the parts 1 and 2.

Art. 55. Again if $(l\,;\, m\,;\, n) = (2\,;\, 2\,;\, \infty)$, we have graphs like

$$\begin{matrix} 2 & 2 \\ 2 & 2 \\ 2 & 1 \\ 1 & 1 \\ 1 & \\ 1 & \end{matrix}$$

We are led to construct the function

$$\frac{1}{(1-ax)\left(1-\dfrac{x}{a}\right)(1-abx^2)\left(1-\dfrac{x^2}{ab}\right)},$$

in the expansion of which all terms involving negative powers of a and b have to be rejected. Isolating the integral portion and putting $a = b = 1$, we find the reduced generating function

$$\frac{1}{(1-x)(1-x^2)^2(1-x^3)}$$

a result which, unlike the previous one, is not obvious.

For the case $(l\,;\, m\,;\, n) = (2\,;\, 3\,;\, \infty)$ we introduce additional denominator factors

$$(1-abcx^3)\left(1-\frac{x^3}{abc}\right),$$

and with increasing labour of algebraical performance we arrive at the reduced generating function

$$\frac{1}{(1-x)(1-x^2)^2(1-x^3)^2(1-x^4)}.$$

Art. 56. In general for the case

$$(l\,;\, m\,;\, n) = (2\,;\, m\,;\, \infty)$$

the generating function is the reciprocal of the product of the $2m$ factors

$$
\begin{array}{c}
(1 - \alpha_1 x)\left(1 - \dfrac{x}{\alpha_1}\right) \\
(1 - \alpha_1\alpha_2 x^2)\left(1 - \dfrac{x^2}{\alpha_1\alpha_2}\right) \\
(1 - \alpha_1\alpha_2\alpha_3 x^3)\left(1 - \dfrac{x^3}{\alpha_1\alpha_2\alpha_3}\right) \\
\vdots \qquad\qquad \vdots \\
(1 - \alpha_1\alpha_2 \ldots \alpha_m x^m)\left(1 - \dfrac{x^m}{\alpha_1\alpha_2 \ldots \alpha_m}\right)
\end{array}
$$

We have to expand this fraction in ascending powers of x, reject all terms containing negative powers of $\alpha_1, \alpha_2, \ldots \alpha_m$, and then obtain the reduced generating function by putting $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 1$ in the portion retained. This has proved a difficult algebraical problem. I am indebted to Professor FORSYTH for a beautiful solution which he will publish elsewhere.* He establishes that the reduced generating function is

$$
\frac{1}{(1-x)\{(1-x^2)(1-x^3)\ldots(1-x^m)\}^2(1-x^{m+1})},
$$

a result which agrees with the prediction. His method is that of selective summation. He forms the general term of the expansion

$$
\left.\begin{array}{l}
\alpha_1^{\,m_1-n_1+m_2-n_2+m_3-n_3+\ldots\ldots} \\
\alpha_2^{\,m_2-n_2+m_3-n_3+\ldots\ldots} \\
\alpha_3^{\,m_3-n_3+\ldots\ldots} \\
\vdots
\end{array}\right\} x^{m_1+n_1+2(m_2+n_2)+3(m_3+n_2)+\ldots\ldots}
$$

and performs the enumeration

$$
\sum_{m_1}\sum_{n_1}\sum_{m_2}\sum_{n_2}\sum_{m_3}\sum_{n_3}\ldots\ldots x^{m_1+n_1+2(m_2+n_2)+3(m_3+n)_3+\ldots\ldots}
$$

for all values of $m_1, n_1, m_2, n_2, m_3, n_3, \ldots\ldots$ which make each of the m expressions

$$
\begin{array}{r}
m_1 - n_1 + m_2 - n_2 + m_3 - n_3 + \ldots\ldots \\
m_2 - n_2 + m_3 - n_3 + \ldots\ldots \\
m_3 - n_2 + \ldots\ldots \\
\ldots\ldots
\end{array}
$$

not less than zero.

The case $(l\,;\, m\,;\, n) = (2\,;\, m\,;\, \infty)$ is thus completely solved by a method which

* 'Proc. L.M.S.,' vol. 27.

appears to be of general application if the difficulties presented by the algebra can be surmounted.

Art. 57. At this point we may enquire into the meaning of the reduced generating function which has been so happily and ingeniously established. We may write it as the product of two fractions :—

$$\frac{1}{(1-x)(1-x^2)\ldots(1-x^m)} + \frac{1}{(1-x^2)(1-x^3)\ldots(1-x^{m+1})}$$

the indication being that every two-layer arrangement is derivable from a combination of two ordinary single-layer partitions whose parts are drawn from the two series of numbers,

$$1, 2, 3, \ldots m,$$
$$2, 3, \ldots m, m+1,$$

respectively. Otherwise we may say that a number N possesses as many two-layer partitions $(2; m; \infty)$ as there are modes of partitionment employing the parts

$$1_1, 2_1, 2_2, 3_1, 3_2, \ldots m_1, m_2, m+1_2.$$

Ex. gr. If $N = 4$ and $m = 3$, the graphs are 9 in number

111	11	11	1	211	21	2	22	2
1	11	1	1		1	1		2
		1	1			1		
			1					

and employing parts

$$\begin{matrix} 1_1 & 2_1 & 3_1 & \\ & 2_2 & 3_2 & 4_2 \end{matrix}$$

we can form 9 partitions, viz. :—

$$(1_1{}^4),\ (2_1 1_1{}^2),\ (2_2 1_1{}^2),\ (2_1{}^2),\ (2_1 2_2),\ (2_2{}^2),\ (3_1 1_1),\ (3_2 1_1),\ (4_2).^*$$

Art. 58. The problem is therefore reduced to establishing a one-to-one correspondence, between the graphs and the partitions of the kind indicated, of general application. I will in part establish this correspondence, which is not very simple in character, later on. At present it is convenient to take a further survey of the general problems in order to obtain ideas concerning the difficulties that confront us.

I form a tableau of algebraic factors.

* The solution thus shows that the two-layer graphs may be exhibited as a one-layer graph by nodes of two colours, say black and red; nodes of different colours not appearing in any single line.

$$(1-p_1x)\left(1-\frac{p_2}{p_1}x\right)\left(1-\frac{p_3}{p_2}x\right)\ldots\left(1-\frac{p_{l-1}}{p_{l-2}}x\right)\left(1-\frac{x}{p_{l-1}}\right),$$

$$(1-p_1q_1x^2)\left(1-\frac{p_2q_2}{p_1q_1}x^2\right)\left(1-\frac{p_3q_3}{p_2q_2}x^2\right)\ldots\left(1-\frac{p_{l-1}q_{l-1}}{p_{l-2}q_{l-2}}x^2\right)\left(1-\frac{x^2}{p_{l-1}q_{l-1}}\right),$$

$$(1-p_1q_1r_1x^3)\left(1-\frac{p_2q_2r_2}{p_1q_1r_1}x^3\right)\left(1-\frac{p_3q_3r_3}{p_2q_2r_2}x^3\right)\ldots\left(1-\frac{p_{l-1}q_{l-1}r_{l-1}}{p_{l-2}q_{l-2}r_{l-2}}x^3\right)\left(1-\frac{x^3}{p_{l-1}q_{l-1}r_{l-1}}\right),$$

$$\text{,,} \qquad \text{,,} \qquad \text{,,} \qquad \ldots \qquad \text{,,} \qquad \text{,,}$$

$$\text{,,} \qquad \text{,,} \qquad \text{,,} \qquad \ldots \qquad \text{,,} \qquad \text{,,}$$

$$(1-p_1q_1\ldots x^m)\left(1-\frac{p_2q_2\ldots}{p_1q_1\ldots}x^m\right)\left(1-\frac{p_3q_3\ldots}{p_2q_2\ldots}x^m\right)\ldots\left(1-\frac{p_{l-1}q_{l-1}\ldots}{p_{l-2}q_{l-2}\ldots}x^m\right)\left(1-\frac{x^m}{p_{l-1}q_{l-1}\ldots}\right),$$

forming a rectangle of m rows and l columns, the letters $p, q, r, \ldots, m$ in number, each occurring with $l-1$ different suffixes.

I say that forming a fraction with unit numerator, having the product of these factors for denominator, we obtain a generating function for the arrangements defined by $(l;\ m;\ \infty)$.

The number of layers is restricted to l (*i.e.*, l or less), and the breadth to m (*i.e.*, m or less), but the graphs are otherwise unrestricted. Reasoning of the same nature as that employed in the simple case of two layers, enables us readily to construct this function. The function is redundant, as we only require that portion of the expansion whose terms are altogether integral. In this portion we put the letters $p, q, r, \ldots$ all equal to unity, and thus arrive at the reduced generating function.

I recall that the predicted result is the reciprocal of

$$\begin{aligned}(1-x)(1-x^2)(1-x^3)\ldots(1-x^{m-2})(1-x^{m-1})(1-x^m)&\\ \times(1-x^2)(1-x^3)\ldots\ldots(1-x^{m-1})(1-x^m)(1-x^{m+1})&\\ \times(1-x^3)\ldots\ldots(1-x^m)(1-x^{m+1})(1-x^{m+2})&\\ \times\ldots&\\ \times(1-x^l)(1-x^{l+1})\ldots\ldots(1-x^{l+m-1}).&\end{aligned}$$

Professor FORSYTH has not yet succeeded in obtaining this result from his powerful method of selective summation. I hear from him that he has verified it in numerous particular cases, but that, so far, he has not been able to surmount the algebraic difficulties presented by the general case.

As regards the final result, the tableau of factors possesses row and column symmetry.

Simple rotation of the graphs through a right angle in the plane xy establishes this intuitively.

We get the same result from m rows and l columns as from l rows and m columns. Taking only the first row, we find that the fraction

$$\frac{1}{(1-p_1x)\left(1-\frac{p_2}{p_1}x\right)\ldots\left(1-\frac{p_{l-1}}{p_{l-2}}x\right)\left(1-\frac{x}{p_{l-1}}\right)}$$

leads to the same reduced generating function as the fraction

$$\frac{1}{(1-p_1x)(1-p_1q_1x^2)\ldots(1-p_1q_1\ldots x^l)},$$

and this is obviously

$$\frac{1}{(1-x)(1-x^2)\ldots(1-x^l)}.$$

Art. 59. If the result predicted be the true result we should be able to establish it by means of a one-to-one correspondence between the graphs and partitions of a certain kind. This presents difficulties to which I will advert in a moment.

Finally I construct the generating function for the case $(l\,;m\,;n)$, the graphs being restricted in all three dimensions.

The numerator is unity and the denominator the product of the factors exhibited in the subjoined tableau :—

$$\begin{array}{ccccc}
(1-gp_1x) & \left(1-\frac{p_2}{p_1}x\right) & \ldots & \left(1-\frac{p_{l-1}}{p_{l-2}}x\right) & \left(1-\frac{x}{p_{l-1}}\right) \\
(1-gp_1q_1x^2) & \left(1-\frac{p_2q_2}{p_1q_1}x^2\right) & \ldots & \left(1-\frac{p_{l-1}q_{l-1}}{p_{l-2}q_{l-2}}x^2\right) & \left(1-\frac{x^2}{p_{l-1}q_{l-1}}\right) \\
,, & ,, & \ldots & ,, & ,, \\
,, & ,, & \ldots & ,, & ,, \\
(1-gp_1q_1\ldots x^m) & \left(1-\frac{p_2q_2\ldots}{p_1q_1\ldots}x^m\right) & \ldots & \left(1-\frac{p_{l-1}q_{l-1}\ldots}{p_{l-2}q_{l-2}\ldots}x^m\right) & \left(1-\frac{x^m}{p_{l-1}q_{l-1}\ldots}\right)
\end{array}$$

in which the occurrence of the symbol g in the first column will be noticed.

We have as usual to neglect all terms in the expansion which involve negative powers of symbols and in addition we must now neglect all terms which involve g raised to a greater power than n.

This construction prevents the lower layer of the graph from having a greater extent than n in the direction Oy, and thus the whole graph is similarly restricted.

The reduced generating functions can be shown in simple instances to agree with the predicted results.

Ex. gr. take $(l\,;\ m\,;\ n) = (2\,;\ 2\,;\ 1)$; the fraction is

$$\frac{1}{(1-gp_1x)\left(1-\dfrac{x}{p_1}\right)(1-gp_1q_1x^2)\left(1-\dfrac{x^2}{p_1q_1}\right)}.$$

We have to retain the integral portion of

$$(1+p_1x+p_1q_1x^2)\left(1+\frac{x}{p_1}+\frac{x^2}{p_1q_1}\right);$$

selecting this and putting $p_1 = q_1 = 1$, we obtain

$$1+x+2x^2+x^3+x^4,$$

which is

$$\frac{(1-x^2)(1-x^3)^2(1-x^4)}{(1-x)(1-x^2)^2(1-x^3)}.$$

As in simpler cases I have not been able to overcome the algebraic difficulties, it is perhaps needless to say that in this most general case I cannot establish the form of the reduced generating function.

Art. 60. I return to consider various particular points of the problem. When the number of layers of nodes is restricted to two, we have seen that the generating function which enumerates the graphs that can be formed with a given number of nodes is

$$(1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-2}(1-x^4)^{-2}\ldots\ldots$$

In correspondence we have the regularised bipartitions (including uni-partitions) of multipartite numbers of given content.

Also if the breadth of the graph do not exceed m *or* the multipartite numbers be not more than m-partite the generating function is

$$(1-x)^{-1}\{(1-x^2)(1-x^3)\ldots(1-x^m)\}^{-2}(1-x^{m+1})^{-1}.$$

I propose to give another proof of these results based upon a certain mode of dissection of the graph.

In the notation that has been used, a graph may be written

$$\begin{array}{cc} 2^{\lambda} & 1^{\mu} \\ 2^{\lambda'} & 1^{\mu'} \\ 2^{\lambda''} & 1^{\mu''} \\ \vdots & \end{array}$$

where

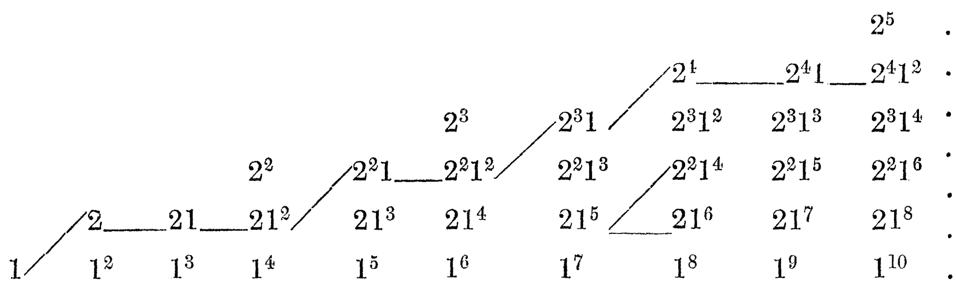
$$\lambda + \mu, \quad \lambda' + \mu', \quad \lambda'' + \mu'', \ldots .$$

and also

$$\lambda, \quad \lambda' \quad \lambda'', \ldots .$$

are in descending order of magnitude.

A line of the graph has a certain weight $2\lambda + \mu$. Any number of lines may be identical and consequently of the same weight, but no two *different* lines may have the same weight in the same graph. Let us form a graph, beginning at the lowest line, taken to be of weight unity, and proceeding upwards through every superior weight. We will find that such a graph may have a variety of forms. Construct the subjoined scheme of graph lines.



In each column every graph line has the same weight. In each line every graph line has the same number of twos. From any graph line, say $2^\lambda 1^\mu$ $(\mu > 0)$, of weight $2\lambda + \mu$, we can pass to a graph line of weight $2\lambda + \mu + 1$ in two ways; viz., by taking $2^\lambda 1^{\mu+1}$ by horizontal progression or $2^{\lambda+1} 1^{\mu-1}$ by diagonal progression. From 2^λ we can only pass to $2^\lambda 1$ by horizontal progression. In accordance with these laws we can form a graph consisting of graph lines of all weights, from unity upwards, in a definite number of ways, depending upon the weight of the highest graph line. For example, we can select the graph whose successive lines are

$$1,\ 2,\ 21,\ 21^2,\ 2^2 1,\ 2^2 1^2,\ 2^3 1,\ 2^4,\ 2^4 1,\ 2^4 1^2 \ldots .$$

The progression from graph line to graph line is either horizontal or diagonal, which we can denote by A and B respectively. Then the graph may be denoted by

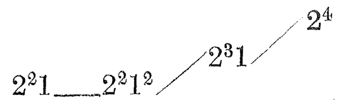
B A A B A B B A A.

The specification of the selected graph may be taken to be a collection of line graphs, each of which is reached by diagonal progression, and which proceeds by horizontal progression.

Thus, in the particular case before us, the specification is

$$2,\ 2^2 1,\ 2^4.$$

To find the form of the graph between any two graph lines, say $2^2 1$, 2^4, we have merely to proceed forwards horizontally from $2^2 1$ and backwards diagonally from 2^4 till a junction is effected in the manner



In the A, B notation the specification is given by the position of the BA contacts.

BAABABBAA.

Every graph that can be selected has its own specification. To enumerate the graphs we must enumerate the specifications.

We must discover the properties of the succession of graph lines which are able to constitute a specification.

If $2^\lambda 1^\mu$ be a graph line in a specification, λ must be greater than zero since the lower line of the scheme cannot be reached by diagonal succession. Also $2^\lambda 1^\mu$ may be followed by a graph line $2^{\lambda_1} 1^{\mu_1}$ so long as $\lambda_1 > \lambda$ and $\lambda_1 + \mu_1 > \lambda + \mu$; and may be preceded by a graph line $2^{\lambda_0} 1^{\mu_0}$ so long as $\lambda > \lambda_0$, $\lambda + \mu > \lambda_0 + \mu_0$.

These laws may be gathered by simple inspection of the scheme.

Hence we may form the graph lines of the specification into a graph of the specification, viz.:—

$$\begin{array}{c} 2^\lambda 1^\mu \\ 2^{\lambda_1} 1^{\mu_1} \\ 2^{\lambda_2 \mu_2} \\ \vdots \end{array}$$

which has the simple properties—

(i) $\lambda > \lambda_1 > \lambda_2 > \dots$

(ii) $\lambda + \mu > \lambda_1 + \mu_1 > \lambda_2 + \mu_2 > \,..$

(iii) The lowest graph line contains, at least, one 2.

Art. 61. Any selected graph has a certain specification graph. From the former we may suppose any graph lines, which do not also belong to the specification, to be absent, and thus obtain a number of graphs which all have the same specification, and may be considered to follow the same line of route through the scheme. Further, the graph lines, whether belonging to the specification or no, may occur any number of times, repeated without the graph of the specification being changed.

Finally, we may associate any partition of a unipartite number with any specification graph whatever, so as to form a two-layer graph. The partition of the unipartite must be interpreted upon that line of route through the scheme which is associated with the given specification graph.

Ex. gr. Suppose the specification graph to be

$$\begin{matrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 1 & \\ 2 & & & \end{matrix}$$

and the unipartite partition to be

$$(9\ 7\ 7\ 6\ 5\ 5\ 4\ 3\ 2\ 2).$$

Interpreted on the line of route concerned, which is that marked upon the scheme above, we obtain

$$\begin{matrix} & 2 & 2 & 2 & 2 & 1 \\ S & 2 & 2 & 2 & 2 & \\ & 2 & 2 & 2 & 1 & \\ & 2 & 2 & 2 & 1 & \\ & 2 & 2 & 1 & 1 & \\ & 2 & 2 & 1 & & \\ & 2 & 2 & 1 & & \\ S & 2 & 2 & 1 & & \\ & 2 & 1 & 1 & & \\ & 2 & 1 & & & \\ & 2 & & & & \\ & 2 & & & & \\ S & 2, & & & & \end{matrix}$$

in which the specification graph lines, marked S, have been interpolated.

Hence, if F (x) be the generating function which enumerates specification graphs,

$$\frac{\mathrm{F}(x)}{(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots \textit{ad inf.}}$$

will be the generating function of all two-layer graphs—that is of forms specified by $(2;\ \infty;\ \infty)$.

We have next to determine the form of F (x).

Art. 62. A specification graph may contain no graph lines; this will be the case when the line of route through the scheme is the lowest horizontal line. There is only *one* such graph; generating function 1. If it contains one line, this line must be of the form $2^{\lambda}1^{\mu}$ $(\lambda > 0)$, and the number of such graphs is given by the generating function

$$\frac{x^2}{(1-x)(1-x^2)}.$$

We may also take the following view of the matter. Let $k_1(x)$ be the generating function for the number of two-layer graphs in which the number of rows is limited to unity; that is, of the graphs specified by (2 ; 1 ; ∞). The graph either does not or does contain a specification graph line. The former are enumerated by

$$\frac{1}{1-x};$$

the latter by $x^2k_1(x)$.

Hence,

$$k_1(x) = \frac{1}{1-x} + x^2k_1(x);$$

or

$$k_1(x) = \frac{1}{(1-x)(1-x^2)};$$

and the number of specification graphs containing one graph line is

$$x^2k_1(x) \quad \text{or} \quad \frac{x^2}{(1-x)(1-x^2)}.$$

Next let $k_2(x)$ denote the number of two-layer graphs in which the number of rows is restricted to two. If it contain no specification graph line it must be of the form

$$\begin{array}{ll} 1^{\lambda} & \\ 1^{\mu} & \quad \lambda \geqq \mu. \end{array}$$

For these the generating function is

$$\frac{1}{(1-x)(1-x^2)}.$$

If it contain *one* specification graph line it must be of the form

$$\begin{array}{l} 2^{\lambda+1}\,1^{\mu} \\ 1^{\nu} \end{array} \quad (\lambda + \mu + 1 \geqq \nu)$$

and these are enumerated by

$$\frac{x^2k_1(x)}{1-x}.$$

If it contain two specification graph lines its form must be

$$\begin{array}{l} 2^{\lambda+2}\,1^{\mu} \\ 2^{\lambda'+1}\,1^{\mu'} \end{array}$$

where

$$\lambda \geqq \lambda' \text{ and } \lambda + \mu + 1 > \lambda' + \mu'.$$

These are enumerated by

$$x^6 k_2(x).$$

Hence

$$k_2(x) = \frac{1}{(1-x)(1-x^2)} + \frac{x^2 k_1(x)}{1-x} + x^6 k_2(x).$$

Therefore

$$k_2(x) = \frac{1}{(1-x)(1-x^2)^2(1-x^3)}$$

and the number of specification graphs containing two graph lines is

$$x^6 k_2(x) \quad \text{or} \quad \frac{x^6}{(1-x)(1-x^2)^2(1-x^3)}.$$

Similarly we shall find that $k_3(x)$ is composed of four parts corresponding to the occurrence of 0, 1, 2, or 3 specification graph lines. The first three are readily seen to be enumerated by generating functions

$$\frac{1}{(1-x)(1-x^2)(1-x^3)}, \quad \frac{x^2 k_1(x)}{(1-x)(1-x^2)}, \quad \frac{x^6 k_2(x)}{1-x};$$

When three specification graph lines occur, the form must be

$$\begin{array}{l} 2^{\lambda+3}\, 1^{\mu} \\ 2^{\lambda'+2}\, 1^{\mu'} \\ 2^{\lambda''+1}\, 1^{\mu''} \end{array}$$

$$\lambda \geqq \lambda' \geqq \lambda'' \text{ and } \lambda + \mu + 2 > \lambda' + \mu' + 1 > \lambda'' + \mu'',$$

and the generating function $x^{12} k_3(x)$.

Hence

$$k_3(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)} + \frac{x^2 k_1(x)}{(1-x)(1-x^2)} + \frac{x^6 k_2(x)}{1-x} + x^{12} k_3(x),$$

and we can show that

$$k_3(x) = \frac{1}{(1-x)(1-x^2)^2(1-x^3)^2(1-x^4)},$$

and the specification graphs containing three graph lines are given by

$$x^{12} k_3(x) \quad \text{or} \quad \frac{x^{12}}{(1-x)(1-x^2)^2(1-x^3)^2(1-x^4)}.$$

In general we obtain the relation

$$k_m(x) = \frac{1}{(1-x)(1-x^2)\ldots(1-x^m)} + \frac{x^2 k_1(x)}{(1-x)(1-x^2)\ldots(1-x^{m-1})}$$

$$+\frac{x^6 k_2(x)}{(1-x)(1-x^2)\ldots(1-x^{m-2})} + \ldots + \frac{x^{s(s+1)} k_s(x)}{(1-x)(1-x^2)\ldots(1-x^{m-s})} + \ldots + x^{m(m+1)} k_m(x),$$

and also

$$k_\infty(x) = \frac{1 + x^2 k_1(x) + x^6 k_2(x) + \ldots + x^{s(s+1)} k_s(x) + \ldots}{(1-x)(1-x^2)(1-x^3)\ldots},$$

where the numerator is the generating function for specification graphs of given content.

Art. 63. We can now establish that $k_s(x)$ is the expression

$$\frac{1}{(1-x)(1-x^2)^2(1-x^3)^2\ldots(1-x^s)^2(1-x^{s+1})};$$

for assume the law true for values of s equal and inferior to $m-1$; substitute in the foregoing identity and writing $1 - x^s = (s)$,

$$(1)(2)\ldots(m)(1-x^{m^2+m})\, k_m(x)$$

$$= 1 + \frac{x^2(m)}{(1)(2)} + \frac{x^6(m)(m-1)}{(1)(2)^2(3)} + \ldots + \frac{x^{m^2-m}(m)(m-1)\ldots(2)}{(1)(2)^2\ldots(m-1)^2(m)}.$$

Recalling the well-known identity

$$\frac{1}{(1-ax)(1-ax^2)\ldots(1-ax^m)} = 1 + \frac{(m)}{(1)}\cdot\frac{ax}{1-ax} + \frac{(m)(m-1)}{(1)(2)}\frac{a^2x^4}{1-ax\,.\,1-ax^2}$$

$$+\frac{(m)(m-1)(m-2)}{(1)(2)(3)}\frac{a^3x^9}{1-ax\,.\,1-ax^2\,.\,1-ax^3} + \ldots$$

and putting therein $a = x$, we find

$$\frac{1}{(2)(3)\ldots(m+1)} = 1 + \frac{x^2(m)}{(1)(2)} + \frac{x^6(m)(m-1)}{(1)(2)^2(3)} + \ldots$$

$$+\frac{x^{m^2-m}(m)(m-1)\ldots(2)}{(1)(2)^2(3)^2\ldots(m-1^2)(m)} + \frac{x^{m^2+m}(m)(m-1)\ldots(1)}{(1)(2)^2(3)^2\ldots(m)^2(m+1)}.$$

Hence

$$(1)(2)\ldots(m)(1 - x^{m^2+m})\, k_m(x) = \frac{1}{(2)(3)\ldots(m+1)} - \frac{x^{m^2+m}}{(2)(3)\ldots(m+1)}.$$

Therefore

$$k_m(x) = \frac{1}{(1)(2)^2(3)^2\ldots(m-1)^2(m)^2(m+1)}.$$

Hence, by induction, it has been established that $k_m(x)$ has this expression for all values of m.

Therefore the result

$$(2\,;\, m\,;\, \infty) = \frac{1}{(1-x)(1-x^2)^2(1-x^3)^2\ldots(1-x^m)^2(1-x^{m+1})}$$

agreeing with that obtained in a totally different manner by FORSYTH.

I hope to continue the theory, adumbrated in this paper, in a future communication to the Royal Society.

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